Static and Dynamic Analysis of a Simple Model of Explicit Gradient Elasticity

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Sideris Stergios – Alexandros

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Kurzfassung

Gradientenelastizität ist ein wichtiger Bereich der modernen Kontinuumsmechanik geworden, mit zahlreichen Anwendungen in Ingenieurwissenschaften in Werkstoffmechanik, experimenteller - und numerischer Mechanik. Die vorliegende Dissertation befasst sich mit einem einfachen Modell der expliziten Gradientenelastizität. Das Ziel ist eine umfassende Untersuchung der Eigenschaften dieses Models. Dazu werden statische und dynamische Probleme mit eindimensionalen und zweidimensionalen (Biege -) Belastungen gelöst. Insbesondere werden eine konsistente Euler - Bernoulli Biegetheorie und verschiedene Versionen des Prinzips von Hamilton benutzt. Ferner wird eine Methode zur Ermittlung der kritischen Last bei Knickung vorgestellt. Die Untersuchungen beleuchten unter Anderem den Einfluss von Randbedingungen und Materialparametern.

Abstract

Gradient elasticity has developed into an important area of continuum mechanics with numerous applications in engineering mechanics, structural analysis, experimental and computational mechanics. The present thesis is concerned with a simple model of explicit gradient elasticity. The aim is to provide a comprehensive insight into the basic properties of this model, by solving several problems in statics and dynamics. The problems include one - dimensional and two - dimensional (bending) loading conditions. Especially, use is made of a consistent Euler - Bernoulli beam theory and of different versions of Hamilton's principle. Moreover, a method is presented for determining the critical load in buckling problems. The investigations highlight, among others, the effect of non - classical boundary conditions and of non - classical material parameters.

Table of contents

1 Introduction - Goal of the Thesis

Gradient effects have been introduced in theories of continuum mechanics since the beginning of the last century. The term "gradient theory" means that besides the state variables, also their higher gradients are taken into account. Thus, for example, in a "first gradient elasticity theory", besides the deformation gradient, which represents a kinematic state variable, its spatial gradient is also taken into account (cf. Mindlin and Eshel [29]). There are different reasons for considering higher gradients in an elasticity theory. Here are some of these reasons listed and at the same time providing the motivation for the present thesis.

In order to model capillarity effects, the "elastic part" of the Korteweg constitutive equation for fluids has taken into account the gradients of density ρ (see Dunn and Serrin [15], where the original work of Korteweg is cited),

$$\mathbf{T} = (-p + \alpha\Delta\rho + \beta|\mathrm{grad}\rho|^2)\mathbf{1} + \delta\mathrm{grad}\rho \otimes \mathrm{grad}\rho + \gamma\mathrm{grad}^2\rho\,. \tag{1.1}$$

In this equation, proposed in 1901, $\mathbf{T}$ is the Cauchy-stress, Δ is the Laplace - Operator, grad is the Gradient - Operator, grad^2 (.) is the second gradient of (.), $\mathbf{1}$ is the unit tensor of second order, $\otimes$ is is the dyadic product and lastly $\alpha,\ \beta,\ \gamma,\ \delta$ are material functions of the density ρ and temperature θ. In order to thermodynamically justify the Korteweg law of elasticity in Eq. (1.1), Dunn and Serrin [15] (see also Dunn [14]) developed non-standard thermodynamics, in which in addition to the classical heat flow, the existence of a mechanical (energy) flow, the so-called "interstitial work flux" is also adopted. However, in the context of Dunn and Serrin's non-standard thermodynamics, it was not possible to justify the term $\gamma\,\mathrm{grad}^2$ in Eq. (1.1). The same applies to the approach of Auffrey et al. [4] and dell'Isola et al. [13]. In addition, Dunn and Serrin have not given precise details on boundary conditions. Nonetheless, the work of Dunn and Serrin can be considered a milestone in the development of the fundamentals of thermodynamics.

Another motivation for the consideration of higher gradients in the theory of elasticity was the modelling of a "microstructure" as an additional "micro-continuum", which is attached at every point of the "macrocontinuum". The first steps in this direction were taken in 1909 by the Cosserat brothers. The further development of this theory by the addition of "micro-inertia terms" is called the micropolar elasticity theory (see Eringen [16], Eringen and Suhubi [17]). In the micropolar continuum, the microcontinuum is a rigid body that can rotate around its centroid. The macrogradient of micropolar rotation appears in theory as a curvature tensor, which causes the existence of so-called "couple stresses" and that the Cauchy stress is not symmetric. The generalization by assuming a homogeneously deformable microcontinuum is due to Eringen [16], Eringen and Suhubi [17] and Mindlin [28] and is known as micromorphic and microstructured elasticity theory, respectively. To the couple stresses in micropolar elasticity correspond so-called hyperstresses (double stresses) in the micromorphic elasticity. In the sixties micromorphic and micropolar elasticity were significantly driven by Mindlin and Eringen (see e.g., Eringen [16], Eringen and Suhubi [17], and Mindlin [28]). Since the micromorphic continuum of Eringen and the microstructured continuum of Mindlin are essentially the same, we will call both micromorphic. Both micropolar and micromorphic elasticity, compared to classical elasticity, contain additional material parameters having the meaning of internal (material) lengths. Two important aspects of both theories are the description of

1. so – called “length scale effects” and

2. dispersion relations.

In particular, the micromorphic elasticity has a remarkably rich spectrum of dispersion relations (see Mindlin [28] and Eringen [16]). Experimental observation of length scale effects regarding bending of beams may be found in Lam et al. [22], while non-classical dispersion relations in conjunction to experimental results are mentioned in Mindlin [28]. In the following, we will call gradient elasticity, an elasticity theory without internal mechanical dissipation, which takes into account state variables and their spatial derivatives, and the Cauchy stress tensor is symmetric. Many gradient elasticity models result directly from the micomorphic elasticity when the micro- and macro- deformation coincide (see Mindlin [28] and Mindlin and Eshel [29]). The resulting theory is called gradient elasticity of the **Toupin - Mindlin - Type**. It is characterized by a free energy per unit volume $\psi = \psi(\boldsymbol{\epsilon}, \mathbf{k})$ and a symmetric Cauchy stress $\boldsymbol{\Sigma}$ which is determined by a spatial Euler - Lagrange derivative

$$\Sigma_{jk} = \frac{\partial \psi}{\partial \epsilon_{jk}} - \frac{\partial}{\partial \mathrm{x}_i}\left(\frac{\partial \psi}{\partial k_{ijk}}\right) . \tag{1.2}$$

In this equation, small deformations are assumed, $\boldsymbol{\epsilon}$ is the strain tensor, $\mathbf{k}$ is the gradient of $\boldsymbol{\epsilon}$ and all components refer to a Cartesian system. (More details about notation are given in the next chapter). It is evident from Eq. (1.2), that $\boldsymbol{\Sigma}$ might be expressed as a function of $\boldsymbol{\epsilon}$ and of higher-order space derivatives of $\boldsymbol{\epsilon}$, including the Laplacian $\Delta\boldsymbol{\epsilon}$ Generally, we shall use here the terminology "Laplacian based explicit gradient elasticity", whenever the Cauchy stress $\boldsymbol{\Sigma}$ may be expressed as a function F of $\boldsymbol{\epsilon}$ and any order Laplacian derivatives of $\boldsymbol{\epsilon}$, i.e.,

$$\boldsymbol{\Sigma} \ = \ F\ (\boldsymbol{\epsilon}, \text{Laplacians of } \boldsymbol{\epsilon})\ . \tag{1.3}$$

Apparently, the most simple constitutive law of the form (1.3), in the case of isotropic material response, reads

$$\boldsymbol{\Sigma} \ = \ \mathbb{C}\,\boldsymbol{\epsilon} - \ l^2\ \mathbb{C}\,\Delta\boldsymbol{\epsilon} \qquad (\textbf{isotropic KG} - \textbf{Model})\ . \tag{1.4}$$

and may be viewed as a so-called standard gradient elasticity model. In Eq. (1.4), l is an internal material length and $\mathbb{C}$ is the fourth-order isotropic elasticity tensor. In the following, we shall call the isotropic constitutive law (1.4) simply as **isotropic KG - Model**. If $\mathbb{C}$ in Eq. (1.4) is replaced by an anisotropic elasticity tensor $\mathbb{K}$ (exhibiting the well - known symmetry conditions), then the resulting elasticity law will be called anisotropic KG - Model,

$$\boldsymbol{\Sigma} \ = \ \mathbb{K}\,\boldsymbol{\epsilon} - \ l^2\ \mathbb{K}\,\Delta\boldsymbol{\epsilon} \qquad (\textbf{anisotropic KG} - \textbf{Model})\ . \tag{1.5}$$

It seems that essentially the constitutive law (1.4) has been introduced for the first time by Altan and Aifantis [3]: These authors employed Eq. (1.4) in order to obtain finite strain field at the crack tip of mode - III crack problems. Later (see Altan and Aifantis [3]), they presented an appropriate energy function leading to this law and provided a general discussion by solving several problems including cracks, propagation of harmonic waves and longitudinal vibrations of a bar. An elaborated discussion of crack problems on the basis of the constitutive law (1.4) may also be found in Georgiadis [18], where Eq. (1.4) and the related field equations and boundary conditions are directly viewed as particular case of Mindlin's gradient elasticity theory. In the context of an analogy between models of gradient elasticity and linear viscoelastic solids, Broese et al. [10] (see also Broese et al. [8, 9]) interpreted Eq. (1.4) as the gradient elasticity counterpart of the Kelvin viscoelastic solid. (The abbreviation KG - Model in Eq. (1.4) stands for Kelvin - Gradient - Elasticity - Model).

It should be noted that a second law of thermodynamics has not been used by Mindlin, while Toupin utilizes a classical second law with non - classical stress power. The non - classical stress power includes terms with hyperstresses, which are conjugate to strain gradients. This approach leads necessarily to the spatial Euler - Lagrange derivative for the Cauchy stress in Eq. (1.2). Obviously, one could ask, why Eq. (1.2) should be the most general constitutive law of gradient elasticity with $\psi = \psi(\boldsymbol{\epsilon}, \mathbf{k})$, especially since the Korteweg law (1.1) is not of the form (1.2). Appart from that, the existence of hyperstresses supposes in general further kinematical degrees of freedom besides the three classical displacement components. The components of higher order gradients of displacement, however, are not new independent degrees of freedom, as for a known displacement field the higher order gradients of the displacement are known too in the interior of the material body. Therefore, one might ask if gradient elastic materials should be considered rather as classical continua in the framework of a non-classical thermodynamics (see Broese et al. [10]). Dunn and Serrin [15] proposed such non-classical thermodynamics which deals with a non-classical energy flux. This thermodynamics however does not produce in essence much more different results in comparison with the approach of Toupin - Mindlin. Similar remarks apply also to the thermodynamics approach of Maugin [26, 27]. Application of the so called "extended irreversible thermodynamics" for a mixture theory with higher-order gradient terms is not known to eliminate the above problems, at least not in the framework of assumptions made in Liu [24, 25]. The non-classical (non-standard or non-conventional) thermodynamics proposed in Alber et al. [1, 2] and Broese et al. [10] might be considered as a further development of the non - classical thermodynamics of Dunn and Serrin and seems to be promising for addressing gradient elasticity problems. On the one hand, this thermodynamics allows to address gradient elasticity models of Non - Toupin - Mindlin - Type with $\psi = \psi(\boldsymbol{\epsilon}, \mathbf{k})$ (see Alber et al. [1]). On the other hand, the Toupin - Mindlin gradient elasticity has been interpreted as a classical continuum with non - classical thermodynamics and non - classical boundary conditions.

The **goal of the thesis** is to provide a comprehensive insight over the basic properties of the KG - Model, by solving several one - and two - dimensional problems in statics and dynamics. In order to assess the capabilities of the KG - Model, we shall compare responses predicted by the KG - Model with corresponding responses predicted by classical elasticity. As far as possible, analytical closed form solutions will be presented. Especially, the scope of the thesis is organised as follows. Section 2 concerns some preliminaries, addressing the notation used in this thesis and the adopted non - conventional thermodynamics. The main issues of the KG - Model are summarized in section 3. Sections 4, 5, 6 are concerned with problems under statical loading conditions, including buckling of columns. The Euler - Bernoulli beam theory adopted in these sections is consistent, i.e., it does not suffer from the well - known inconsistency between the field equations and elasticity law. Dynamical problems are discussed in sections 7, 8, 9. The discussions rely upon governing equations which are established by employing appropriate versions of Hamilton's principle. Conclusions which may be drawn from the analysis provided in the thesis are stated in section 10. The main difference between the present work and similar studies on this topic is the discussion of limiting cases in the material response, by considering specific boundary conditions, by interpreting the results in the setting of classical continua and by deriving a consistent Euler - Bernoulli beam theory.

2 Preliminaries

2.1 Notation

In order to facilitate comparison with the works of Mindlin [28] (see also Mindlin and Eshel [29]), we largely use the same notation as in these works.

Since the analysis in this thesis is related to the works of Mindlin, the same notation will be used. The deformations are assumed to be small. For that reason, no distinction between reference and actual configuration will be made. In addition, all processes are assumed to be isothermal, all indices have the range of integers (1, 2, 3) and summation over repeated indices is implied. Often, a tensor **A** will be identified by its components $\mathrm{A}_{i\ldots j}$. All tensorial components will refer to a Cartesian coordinate system x_i on a 3 - dimensional Euclidean space, which induces the orthonormal basis $\{\mathrm{e}_i\}$.

We write $\mathrm{A}_{i\ldots(jk)..p}$ and $\mathrm{A}_{i\ldots[kj]\ldots p}$ for the symmetric and the skew-symmetric part of **A** with respect to the indices j and k, respectively. Thus, if **A** is a second-order tensor with components A_{ij}, then $\mathrm{A}_{(ij)}$ and $\mathrm{A}_{[ij]}$ are the components of its symmetric part $\mathrm{A}^{(S)}$ and its skew-symmetric part $\mathrm{A}^{(A)}$, i.e., $A_{(ij)} \equiv A_{ij}^{(S)}$ and $A_{[ij]} \equiv A_{ij}^{(A)}$. For vectors **a**, second - order tensors **A** and fourth - order tensors $\mathbb{K}$ we have $\mathrm{a} = \mathrm{a}_i\, \mathbf{e}_i$, $\mathrm{A} = \mathrm{A}_{ij}\, \mathbf{e_i} \otimes \mathbf{e}_j$, $\mathbb{K} = \mathbb{K}_{ijmn}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_m \otimes \mathbf{e}_n$, where $\otimes$ is the tensorial product. The unit second - order tensor **1** has the representation $\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, where δ_{ij} is the Kronecker - delta. The scalar product $\mathbf{a} \cdot \mathbf{b}$ between vectors **a** and **b** can be expressed as $\mathbf{a} \cdot \mathbf{b} = \mathrm{a}_i \mathrm{b}_i$.

In the remainder of the thesis, we denote by $\mathbb{C}$ a fourth - order, symmetric, isotropic, elasticity tensor,

$$\mathbb{C}_{ijmn} = \lambda^* \delta_{ij}\delta_{mn} + \mu^*(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \,, \tag{2.1}$$

where λ^*, μ^* are Lamé moduli and $\mu^* > 0$, $2\mu^* + 3\lambda^* > 0$. Between λ^*, μ^*, Young's modulus E and Poisson ratio ν, there are the well - known relations

$$\lambda^* = \frac{E\nu}{(1+\nu)(1-2\nu)} \,, \qquad \mu^* = \frac{E}{2(1+\nu)} \,. \tag{2.2}$$

Explicit reference to space and time variables, upon which a function depends, will be dropped in most parts of the thesis. Also, we shall not distinguish between functions and their values. However, to make things clear, when necessary, we shall give explicitly the set of variables which the function depends upon. Moreover, let $\mathfrak{B}$ be a material body, which may be identified by the position vectors $\mathrm{x} = \mathrm{x}_i \mathrm{e}_i$ and which occupies the space V in the three - dimensional Euclidean point space we deal with. We indicate by **n** the outward unit normal vector to the surface ∂V bounding the space V. If f is a function of position **x** and time t, then we use the notations

$$\dot{f} \coloneqq \frac{\partial f}{\partial t} \,, \qquad \partial_i f \coloneqq \frac{\partial f}{\partial \mathrm{x}_i} = f_{,\mathrm{x}_i} \,. \tag{2.3}$$

The gradient of f is denoted by $\mathrm{grad} f \equiv \nabla f$, so that, e.g., for a second-order tensor **A**, we have

$$(\nabla A)_{ijk} = \partial_i A_{jk} \,. \tag{2.4}$$

The divergence operator is indicated by div, so that, e.g., for a third-order tensor $\boldsymbol{\mu}$, $\mathrm{div}\boldsymbol{\mu}$ is a second-order tensor with components

$$(\mathrm{div}\mu)_{\mathrm{jk}} = \partial_i \mu_{ijk} \,. \tag{2.5}$$

The Laplacian of f, written Δf, is defined through

$$\Delta f = \mathrm{divgrad} f \equiv \partial_i \partial_i f. \tag{2.6}$$

Finally, for a function $f(\mathbf{x}, t)$, with $\mathrm{x} \in V \cup \partial V$, the normal derivative $Df(\mathrm{x}, t)$ and the surface derivative $D_i f(\mathrm{x}, t)$ are defined by

$$Df \coloneqq \mathrm{n}_l \partial_l f \,, \tag{2.7}$$

$$D_i f \coloneqq \partial_i f - \mathrm{n}_i Df \,, \tag{2.8}$$

for every $\mathrm{x} \in \partial V$ (see Mindlin [28], p. 401, or Mindlin and Eshel [29], p. 112). Using the notation introduced above, and writing $\mathbf{u}$ for the displacement vector, we have

$$\epsilon_{ij} \equiv \epsilon_{(ij)} \coloneqq \frac{1}{2}\left(\partial_i \mathrm{u}_j + \partial_j \mathrm{u}_i\right) = \partial_{(i} \mathrm{u}_{j)} \,, \tag{2.9}$$

$$k_{ijk} \equiv k_{i(jk)} \coloneqq \partial_i \epsilon_{jk} = \partial_i \partial_{(j} \mathrm{u}_{k)} \,. \tag{2.10}$$

We will be concerned with material bodies governed by ordinary balance laws of linear and angular momentum leading to the local forms

$$\partial_j \Sigma_{jk} + F_k + I_k = 0 \,, \quad \Sigma_{jk} = \Sigma_{kj} \,, \tag{2.11}$$

where, $\mathbf{\Sigma}$ is the Cauchy stress tensor. The body force and the inertial force vectors are denoted by $\mathbf{F}$ and $\mathbf{I}$, with components F_k and I_k, respectively. The two forces $\mathbf{F}$ and $\mathbf{I}$ may be composed respectively of classical and non - classical parts. If we define a generalized body force $\mathbf{b}$ through

$$\mathbf{b} \coloneqq \mathbf{F} + \mathbf{I} \,, \tag{2.12}$$

then Eq. (2.11) takes the form

$$\partial_j \Sigma_{jk} + b_k = 0 \,. \tag{2.13}$$

It is perhaps of interest to make some comments on the generalized force $\mathbf{b}$. In classical mechanics, no parts of force $\mathbf{b}$ are present in boundary conditions. However, in gradient elasticity, non-classical terms may be involved in force $\mathbf{b}$, which can be assumed both to contribute or not to boundary conditions (see Mindlin [28] and Broese et al. [8, 9]). Such aspects are discussed in detail in section 7. When non - classical terms are not present, then Eq. (2.11) takes the form

$$\partial_j \Sigma_{jk} + F_k - \rho \ddot{\mathrm{u}}_k = 0 \,, \tag{2.14}$$

with ρ being the mass density and F_k being components of classical body force.

2.2 Non - conventional thermodynamics framework

This part of the thesis is taken from Alber et al. [2] and Broese et al. [10].

It is assumed that radiant heating, chemical reactions, and electromagnetic effects are absent. Then, the local form of the energy balance for material body $\mathfrak{B}$ (first law of thermodynamics) reads

$$\dot{e} = w_{st} - \partial_i q_i \ , \tag{2.15}$$

where e is the internal energy measured per unit volume, w_{st} is the stress power per unit volume and $\mathbf{q}$ is the energy/heat flux vector. Toupin [32, 33] suggested the possibility for $\mathbf{q}$ to encapsulate more than heat flux, and this has been in fact elaborated by Dunn and Serrin [15] and Dunn [14].

Fundamental in usual irreversible thermodynamics is the hypothesis of a local equilibrium state. It assumes that each material point of $\mathfrak{B}$ behaves like a simple homogenous system in equilibrium, so that absolute temperature $\theta > 0$ and entropy per unit volume η may be assigned to that point. The free energy per unit volume is defined through

$$\psi \coloneqq e - \theta\eta \ , \tag{2.16}$$

and the energy law (2.15) takes the equivalent form

$$w_{st} - \dot{\psi} - \theta\dot{\eta} - \eta\dot{\theta} - \partial_i q_i = 0 \ . \tag{2.17}$$

The second law of thermodynamics is commonly accepted in the form of the Clausius - Duhem inequality, stating that the entropy production γ is non - negative,

$$\gamma \coloneqq \dot{\eta} + \partial_i \left(\frac{q_i}{\theta}\right) \geq 0 \ . \tag{2.18}$$

Now consider a class of materials that are sensitive to non-localities in space effects. For example, assume the free energy function ψ to depend, besides on state variables permitted in classical irreversible thermodynamics, also on the spatial gradients of these variables. Since gradient terms indicate neighbourhood effects, the hypothesis of a classical local equilibrium state is generally no longer justified. Yet, according to classical irreversible thermodynamics, absolute temperature θ and entropy η can be attributed only to equilibrium states. We may proceed conceptually further along the lines of classical irreversible thermodynamics as follows.

The state of each material point of $\mathfrak{B}$ is assumed at any time to be associated with a homogenous material system in equilibrium, which we call the generalized associated local equilibrium state or system. Classical thermostatics ensures for the generalized associated equilibrium system the existence of absolute temperature $\theta(\mathrm{x}, t)$ and entropy $\eta(\mathrm{x}, t)$, and these are attributed to be the temperature and entropy of the real material at (x, t). Denote by υ_I, I = 1,...,N_I, the components of state variables and by ξ_J, J = 1,...,N_J, components of time and space derivatives of υ_I, and assume for the real material

$$\psi = \psi\big(\upsilon_I, \xi_J, \theta\big) \ . \tag{2.19}$$

We generally call such functions, as ψ ( ) on the right - hand side of Eq. (2.19), as response functions. For the purposes of the present thesis, it suffices to suppose that time and space derivatives of θ are not included in ξ_J .

The mass density and the response function of free energy of the generally fictitious local equilibrium system are defined to be the same as for the real material characterized by Eq. (2.19). For both, the real material and the associated equilibrium state, the free energy ψ and the internal energy e are postulated to satisfy Eq. (2.16). In other words, not only the free energy but also the internal energy is identical in the two systems. Let w_{st} be the stress power and $\mathbf{q}$ the energy/heat flux vector for the real material, so that the energy balance laws (2.15) and (2.17) hold for the real material. We will introduce an energy balance for the generalized associated equilibrium system by regarding ξ_J for this (homogenous) system as new state variables, which are independent of υ_I, θ. For example, assume ϵ_{ij} and $\partial_k \epsilon_{ij}$ to be included as state variables in the response function of ψ, where ϵ_{ij} are the components of the infinitesimal strain tensor $\boldsymbol{\epsilon}$. Then, $\partial_k \epsilon_{ij}$ have to be regarded for the generalized associated local equilibrium state as new, independent kinematical variables. These, again, engender additional, higher-order stresses and hence the stress power $\overline{w}_{st}$ entering into the energy balance law for the fictitious generalized associated local equilibrium state, will be in general different from w_{st}. We denote by $\partial_i \overline{q}_i$ the energy/heat flux supply for the associated equilibrium system and postulate for this system the energy balance law

$$\dot{e} = \overline{w}_{st} - \partial_i \overline{q}_i \quad \Leftrightarrow \quad \overline{w}_{st} - \dot{\psi} - \theta \dot{\eta} - \eta \dot{\theta} - \partial_i \overline{q}_i = 0 \,. \tag{2.20}$$

Next define stress power w'_{st} and energy/heat flux q' through

$$w'_{st} \coloneqq w_{st} - \overline{w}_{st} \,, \tag{2.21}$$

$$\mathbf{q}' \coloneqq \mathbf{q} - \overline{\mathbf{q}} \,, \tag{2.22}$$

so that, from Eqs. (2.15) and (2.20),

$$w'_{st} = \partial_i q'_i \,. \tag{2.23}$$

Further, assume that w'_{st} and $\mathbf{q}'$ can be decomposed in N parts $w'_{st(i)}$ and $\mathbf{q}'_{(i)}$,

$$w'_{st} = w'_{st(1)} + w'_{st(2)} + \cdots + w'_{st(N)} \,, \tag{2.24}$$

$$\mathbf{q}' = \mathbf{q}'_{(1)} + \mathbf{q}'_{(2)} + \ldots + \mathbf{q}'_{(N)} \,, \tag{2.25}$$

and postulate energy/heat transfer into mechanical power through

$$\begin{aligned} w'_{st(1)} &= \partial_i q'_{(1)i} \\ w'_{st(2)} &= \partial_i q'_{(2)i} \\ &\;\;\cdots\cdots \\ w'_{st(N)} &= \partial_i q'_{(N)i} \end{aligned} \tag{2.26}$$

In order to complete the theory, some constitutive equations for $w'_{st(i)}$ and $\mathbf{q}'$ remain to be specified. By doing so, it might be that new variables will be involved.

The physical idea behind these equations is that the energy/heat flux difference $\mathbf{q}'$, between the actual and the generalized local equilibrium state, may be composed of various parts, say N, which can be related to corresponding energy carriers. These carriers provide the opportunity for producing some energy/heat transfer to mechanical power without affecting the internal energy, as manifested by Eqs. (2.21) - (2.26). The assumed transfer must be accounted for in the entropy production and hence it is postulated that

$$\overline{\gamma} \coloneqq \dot{\eta} + \partial_i \left(\frac{\overline{q}_i}{\theta} \right) \geq 0 \,, \tag{2.27}$$

or, in view of Eq. (2.20),

$$-\left(\eta\dot{\theta}+\dot{\psi}\right)+\overline{w}_{st}-\frac{1}{\theta}\overline{q}_i\partial_i\theta\geq 0\,. \tag{2.28}$$

Inequality (2.27) (respectively (2.28)) is the Clausius - Duhem inequality for the generalized associated local equilibrium state, which is supposed to apply for the real material as well. Once more, it is worth remarking, that the concept of the generalized associated local equilibrium state imposes the existence of absolute temperature and entropy and motivates the introduction of inequality (2.27) or (2.28). Otherwise, these inequalities can be exploited by employing known methods in continuum thermodynamics. That means, like classical irreversible thermodynamics based on the hypothesis of a local equilibrium state, when exploiting the inequality, temporal and space derivatives of the state variables can be elaborated. As mentioned above, throughout the present thesis we assume that all response functions do not depend on time and space derivatives of θ. Then, by using the Coleman - Noll procedure [11, 12], it can be proved that

$$\eta=-\frac{\partial\psi\left(\upsilon_I,\xi_J,\theta\right)}{\partial\theta}\,. \tag{2.29}$$

This potential relation will hold in the remainder of the present thesis. Concluding, we would like to remark that the main difference of this thermodynamic approach to other approaches on the same subject is the energy transfer Eqs. (2.21) - (2.26), which are not postulated in other theories. Practically, these energy transfer equations make possible it to associate the real material system with the fictitious local equilibrium system.

3 Governing equations and concomitant boundary conditions without non - classical acceleration terms

In section 3, we summarize the governing equations for the KG - Model, in the framework of the non - conventional thermodynamics of Alber et al. [2] and Broese et al. [8, 9] (see also section 2.2). For simplicity, non - classical body and non - classical acceleration forces are excluded here.

3.1 Explicit gradient elasticity - basic assumptions

Following Broese et al. [8, 9], we focus attention on material bodies which obey the ordinary balance law (2.14). After multiplying this by $\dot{\mathrm{u}}_k$, integrating over V, keeping in mind Eq. (2.9), and using standard steps, we arrive at

$$\int_{\partial V} \mathrm{n}_j \Sigma_{jk} \dot{\mathrm{u}}_k \, dS + \int_V F_k \dot{\mathrm{u}}_k \, dV = \int_V \Sigma_{jk} \, \dot{\epsilon}_{jk} dV + \int_V \rho \, \ddot{\mathrm{u}}_k \dot{\mathrm{u}}_k \, . \tag{3.1}$$

From the right hand side of this equation, we recognize that the term

$$w_{st} = \Sigma_{jk} \dot{\epsilon}_{jk} \tag{3.2}$$

Represents the classical stress power.

Now assume for the general non - isothermal case that the response function of the free energy ψ has the form (explicit gradient elasticity)

$$\psi = \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta) \, . \tag{3.3}$$

Suppose that w_{st} and $\mathbf{q}$ satisfy Eqs. (2.20) - (2.26), with $N = 1$

$$w_{st} = \overline{w}_{st} + w'_{st} \, , \qquad \mathbf{q} = \overline{\mathbf{q}} + \mathbf{q}' \, , \tag{3.4}$$

and

$$w_{st} - w'_{st} - \dot{\psi} - \theta \dot{\eta} - \eta \dot{\theta} - \partial_i \overline{q}_i = 0, \tag{3.5}$$

$$w'_{st} = \partial_i q'_i \, . \tag{3.6}$$

It follows from Eq. (2.28) that

$$\Sigma_{jk} \dot{\epsilon}_{jk} - w'_{st} - \eta \dot{\theta} - \dot{\psi} - \frac{1}{\theta} \overline{q}_i \partial_i \theta \geq 0 \, . \tag{3.7}$$

In view of Eq. (2.29), we have on the one hand

$$\eta = - \frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial \theta} \, . \tag{3.8}$$

On the other hand, it is convenient to introduce the potential relations

$$\tau_{jk} \coloneqq \frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial \epsilon_{jk}} \, , \tag{3.9}$$

$$\mu_{ijk} := \frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial k_{ijk}} \,. \tag{3.10}$$

Then, inequality (3.7) furnishes

$$\left(\Sigma_{jk} - \frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial \epsilon_{jk}}\right)\dot{\epsilon}_{jk} - \frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial k_{ijk}}\dot{k}_{ijk} - w'_{st} - \frac{1}{\theta}\overline{q}_i \partial_i \theta \geq 0 \,, \tag{3.11}$$

or equivalently,

$$\left(\Sigma_{jk} - \tau_{jk}\right)\dot{\epsilon}_{jk} - \mu_{ijk}\dot{k}_{ijk} - w'_{st} - \frac{1}{\theta}\overline{q}_i \partial_i \theta \geq 0 \,. \tag{3.12}$$

In what follows, we specify the constitutive structure by assuming response functions

$$\Sigma_{jk} = \Sigma_{jk}(\boldsymbol{\epsilon}, \mathbf{k}, \theta) \,, \tag{3.13}$$

$$\overline{q}_i = \overline{q}_i(\boldsymbol{\epsilon}, \mathbf{k}, \theta, \nabla\theta) \,. \tag{3.14}$$

It is readily seen that the identity

$$\mu_{ijk}\dot{k}_{ijk} = \mu_{ijk}\partial_i \dot{\epsilon}_{jk} = \partial_i\left(\mu_{ijk}\dot{\epsilon}_{jk}\right) - \left(\partial_i \mu_{ijk}\right)\dot{\epsilon}_{jk} \tag{3.15}$$

applies, so that, from inequality (3.12),

$$\left(\Sigma_{jk} - \tau_{jk} + \partial_i \mu_{ijk}\right)\dot{\epsilon}_{jk} - \partial_i\left(\mu_{ijk}\dot{\epsilon}_{jk}\right) - w'_{st} - \frac{1}{\theta}\overline{q}_i \partial_i \theta \geq 0 \,. \tag{3.16}$$

A simple way to always fulfil this inequality is, first, to make the constitutive assumption

$$w'_{st} := -\partial_i\left(\mu_{ijk}\dot{\epsilon}_{jk}\right) , \tag{3.17}$$

so that inequality (3.12) becomes

$$\left(\Sigma_{jk} - \tau_{jk} - \partial_i \mu_{ijk}\right)\dot{\epsilon}_{jk} - \frac{1}{\theta}\overline{q}_i \partial_i \theta \geq 0 \,. \tag{3.18}$$

Keeping in mind Eqs.(3.13), (3.14) and using Coleman - Noll's arguments [11, 12], we can prove that the relations

$$\Sigma_{jk} = \tau_{jk} - \partial_i \mu_{ijk} = \frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial \epsilon_{jk}} - \partial_i\left(\frac{\partial \psi(\boldsymbol{\epsilon}, \mathbf{k}, \theta)}{\partial k_{ijk}}\right) , \tag{3.19}$$

$$\overline{q}_i \partial_i \theta \leq 0 \tag{3.20}$$

are necessary and sufficient conditions for inequality (3.18) to hold always. It is worth remarking that Eq. 3.19) is nothing but the spatial Euler - Lagrange derivative in Eq. (1.2).

Equation (3.6) can be satisfied trivially by the constitutive assumption

$$q'_i = -\mu_{ijk}\dot{\epsilon}_{jk} + c_i \,, \qquad \partial_i c_i = 0 \,. \tag{3.21}$$

For simplicity, the divergence - free vector $\mathbf{c}$ is assumed to vanish in the following, $\mathbf{c} = \mathbf{0}$. That way, we may conclude from Eqs. (3.5), (3.6), that

$$\Sigma_{jk}\dot{\epsilon}_{jk} - \partial_i q_i' - \dot{\psi} - \theta\dot{\eta} - \dot{\theta}\eta - \partial_i \overline{q}_i = 0 \,. \tag{3.22}$$

On the basis of this energy law, we define the fully recoverable isothermal case through $\theta = \theta_0 = \text{const.}$ and

$$\theta\dot{\eta} + \partial_i \overline{q}_i = 0 \,, \qquad \psi = \psi(\boldsymbol{\epsilon}, \mathbf{k}) \,. \tag{3.23}$$

It follows from Eq. (3.22), that

$$\frac{d}{dt}\psi(\boldsymbol{\epsilon}, \mathbf{k}) = \Sigma_{jk}\dot{\epsilon}_{jk} - \partial_i q_i' = \overline{w}_{st} \,, \tag{3.24}$$

where, Eqs. (3.2), (3.4), (3.6) have been taken into account. Evidently, the isothermal version of Eq. (3.19) is

$$\Sigma_{jk} = \tau_{jk} - \partial_i \mu_{ijk} \quad \Leftrightarrow \quad \boldsymbol{\Sigma} = \boldsymbol{\tau} - \mathrm{div}\boldsymbol{\mu} \,, \tag{3.25}$$

with

$$\tau_{jk} = \frac{\partial\psi(\boldsymbol{\epsilon}, \mathbf{k})}{\partial\epsilon_{jk}} \,, \tag{3.26}$$

$$\mu_{ijk} = \mu_{i(jk)} = \frac{\partial\psi(\boldsymbol{\epsilon}, \mathbf{k})}{\partial_i\epsilon_{jk}} \,. \tag{3.27}$$

Eq.(3.25) allows to rewrite the balance law (2.14) as

$$\partial_j\left(\tau_{jk} - \partial_i \mu_{ijk}\right) + F_k - \rho\ddot{u}_k = 0 \,. \tag{3.28}$$

It is of interest to remark, that Eq. (3.25) represents a constitutive law for the Cauchy stress tensor $\boldsymbol{\Sigma}$ suggested by the thermodynamical restrictions. The stresses $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ have to be regarded as internal state variables, which are determined by the deformation as stated in Eqs. (3.26), (3.27).

3.2 The KG – Model in the framework of non – conventional thermodynamics

We can now establish the isotropic KG - Model in Eq. (1.4) by assuming the response function for ψ to have the form (see Altan and Aifantis [3] and Georgiadis [18])

$$\psi = \frac{1}{2}\epsilon_{ij}\mathbb{C}_{ijmn}\epsilon_{mn} + \frac{l^2}{2}k_{ijk}\mathbb{C}_{jkmn}k_{imn} \,, \tag{3.29}$$

where l is an internal material length (see Introduction 1). From Eqs. (3.29), (3.26), (3.27),

$$\tau_{jk} = \mathbb{C}_{ijmn}\epsilon_{mn} \,, \tag{3.30}$$

$$\mu_{ijk} = l^2\mathbb{C}_{jkmn}k_{imn} = l^2\mathbb{C}_{jkmn}\partial_i\epsilon_{mn} \,, \tag{3.31}$$

and from Eq. (3.25),

$$\Sigma_{jk} = \mathbb{C}_{jkmn}\epsilon_{mn} - l^2\mathbb{C}_{jkmn}\partial_i\partial_i\epsilon_{mn} \qquad \Leftrightarrow \tag{3.32}$$

$$\boldsymbol{\Sigma} = \boldsymbol{\tau} - \mathrm{div}\boldsymbol{\mu} = \mathbb{C}\boldsymbol{\epsilon} - l^2\mathbb{C}\Delta\boldsymbol{\epsilon} \quad \text{(isotropic KG - Model)}\,. \tag{3.33}$$

The constitutive law (3.33) is nothing but the isotropic KG - Model in Eq. (1.4). Note, that the anisotropic KG - Model (cf. the remarks on p. 2) is obtained from the above relations by replacing the isotropic elasticity tensor $\mathbb{C}$ with an anisotropic one, $\mathbb{K}$ (see Eq. (1.5)). If not stated otherwise, all discussions will be concerned with the isotropic KG - Model. Only when formulating a consistent Euler - Bernoulli beam theory, we shall make use of the anisotropic KG - Model.

Figure 1 displays the mechanical analog of the constitutive law (3.25), respectively (3.33). It is the gradient elasticity counterpart of the Kelvin model in linear viscoelasticity and consists of two springs in parallel. One spring (corresponding to Eq. (3.26), respectively (3.30)) is standard, i.e., its response obeys a classical elasticity law. The other (corresponding to Eq. (3.27), respectively (3.31)), is a non - standard spring, which acts only if the strain is non - homogenous. These issues justify the name KG - Model for the constitutive law (3.33).

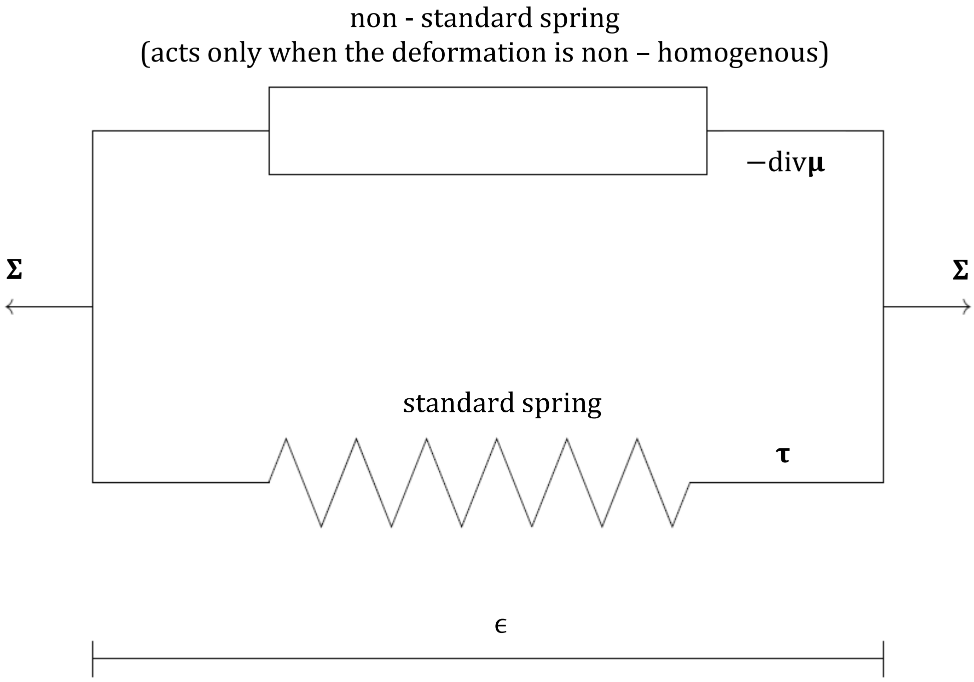


Figure 1: The gradient elasticity Kelvin model (KG - Model): Two springs in parallel.

To accomplish the theory, we have to formulate appropriate boundary conditions. Note that such boundary conditions cannot be assumed arbitrarily. Any elasticity theory, and therefore gradient elasticity too, is postulated to be a theory without internal mechanical dissipation (see Introduction 1). In other words, the power of the external forces should be equal to the rate of kinetical energy and the power stored in the material as rate of the free energy ψ. This fact suggests the way the required boundary conditions have to be obtained. To see this, we start by taking the volume integral of Eq. (3.24), keeping in mind Eq. (3.1),

$$\int_{\partial V} \mathrm{n}_j \Sigma_{jk} \dot{\mathrm{u}}_k \, dS + \int_V F_k \dot{\mathrm{u}}_k \, dV - \int_V \rho \ddot{\mathrm{u}}_k \dot{\mathrm{u}}_k \, dV - \int_{\partial V} \mathrm{n}_i q_i' dS = \frac{d}{dt}\int_V \psi(\boldsymbol{\epsilon}, \mathbf{k}) dV\,, \tag{3.34}$$

where the divergence theorem has been applied to transform a volume integral into a surface integral (the fourth integral on the left - hand side). By recalling Eqs. (3.33) and (3.21), we may infer from Eq. (3.34) that

$$\int_{\partial V} \mathrm{n}_j\big(\tau_{jk} - \partial_i\mu_{ijk}\big)\dot{\mathrm{u}}_k\, dS + \int_{\partial V} \mathrm{n}_i\mu_{ijk}\,\partial_j\dot{\mathrm{u}}_k\, dS + \int_V \mathrm{F}_k\dot{\mathrm{u}}_k\, dV = \frac{d}{dt}\int_V \psi(\boldsymbol{\epsilon},\mathbf{k})dV + \int_V \rho\ddot{\mathrm{u}}_k\dot{\mathrm{u}}_k\, dV\,. \tag{3.35}$$

The second integral on the left-hand side must be resolved further since the rate $\partial_j\dot{\mathrm{u}}_k$ is not independant of $\dot{\mathrm{u}}_k$ on ∂V. By using known results of vector and tensor analysis (see Brand [6], p. 222 and Broese et al. [8, 9]), the identity

$$\int_{\partial V} \mathrm{n}_i\mu_{ijk}\,\partial_j\dot{\mathrm{u}}_k\, dS = \int_{\partial V} \mathrm{n}_i\mathrm{n}_j\mu_{ijk}\, D\dot{\mathrm{u}}_k\, dS + \int_{\partial V}\big[(D_l\mathrm{n}_l)\mathrm{n}_i\mathrm{n}_j\mu_{ijk} - D_j\big(\mathrm{n}_i\mu_{ijk}\big)\big]\dot{\mathrm{u}}_k\, dS \tag{3.36}$$

can be established. After substituting into Eq. (3.35),

$$\begin{aligned}\int_{\partial V}\big[\mathrm{n}_j\big(\tau_{jk} - \partial_i\mu_{ijk}\big) - D_j\big(\mathrm{n}_i\mu_{ijk}\big) + (D_l\mathrm{n}_l)\mathrm{n}_i\mathrm{n}_j\mu_{ijk}\big]\dot{\mathrm{u}}_k\, dS + \int_{\partial V}\mathrm{n}_i\mathrm{n}_j\mu_{ijk}\, D\dot{\mathrm{u}}_k\, dS + \int_V \mathrm{F}_k\dot{\mathrm{u}}_k\, dV \\ = \frac{d}{dt}\frac{1}{2}\int_V \rho\dot{\mathrm{u}}_k\dot{\mathrm{u}}_k\, dV + \frac{d}{dt}\int_V\psi(\boldsymbol{\epsilon},\mathbf{k})dV = \frac{d}{dt}\left[\frac{1}{2}\int_V\rho\dot{\mathrm{u}}_k\dot{\mathrm{u}}_k\, dV + \int_V\psi(\boldsymbol{\epsilon},\mathbf{k})dV\right].\end{aligned} \tag{3.37}$$

The left - hand side represents the power expended by external contact forces (corresponding to the surface integral) and by external body forces (corresponding to the volume integral). The first integral on the far right-hand side is the kinetic energy of the body, while the second integral represents the total energy stored in the material.

Since the rates $\dot{\mathrm{u}}_k$ and $\mathrm{D}\dot{\mathrm{u}}_k$ are independent of each other on ∂V, the two surface integrals in Eq. (3.37) suggest the following concomitant boundary conditions:

$$\text{either } P_k \text{ or } \mathrm{u}_k \quad \text{(classical boundary conditions)}\,, \tag{3.38}$$

$$\text{and either } R_k \text{ or } D\mathrm{u}_k \quad \text{(non - classical boundary conditions)} \tag{3.39}$$

have to be prescribed on ∂V, where the classical traction $\mathbf{P}$ and the non - classical traction $\mathbf{R}$ are defined through

$$P_k \coloneqq \mathrm{n}_j\big(\tau_{jk} - \partial_i\mu_{ijk}\big) - D_j\big(\mathrm{n}_i\mu_{ijk}\big) + (D_l\mathrm{n}_l)\mathrm{n}_i\mathrm{n}_j\mu_{ijk}\,, \tag{3.40}$$

$$R_k \coloneqq \mathrm{n}_i\mathrm{n}_j\mu_{ijk}\,. \tag{3.41}$$

These definitions allow to rewrite Eq. (3.37) as

$$\int_{\partial V}(\mathbf{P}\cdot\dot{\mathbf{u}} + \mathbf{R}\cdot D\dot{\mathbf{u}})dS + \int_V \mathbf{F}\cdot\dot{\mathbf{u}}\, dV = \frac{d}{dt}\frac{1}{2}\int_V\rho\,\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\, dV + \frac{d}{dt}\int_V\psi dV\,, \tag{3.42}$$

According to the non - classical thermodynamics approach, the classical boundary conditions have to be thought mainly to correspond to the required boundary conditions for the balance law (2.14). The non - classical boundary conditions have to be viewed essentially as constitutive boundary conditions in connection with the constitutive law (3.33). In the sense of the analogy to viscoelasticity, the constitutive boundary conditions in gradient elasticity are the counterpart of the constitutive initial conditions in viscoelasticity.

4 One - dimensional problems in statics

Two one - dimensional static problems will be discussed in this section. The first one is a bar under static uniaxial tensile loading and the second is extension of a bar under its own weight.

4.1 Uniaxial tensile loading of a bar

4.1.1 Responses predicted by the isotropic KG - Model for homogenous non - classical boundary conditions

Consider a bar with length L, sectional area A and Poisson ratio $\nu = 0$, implying where $\lambda^* = 0$ and $2\mu^* = E$. The bar is subject to one - dimensional loading conditions with $\mathrm{x}_1 = \mathrm{x}$, $\mathrm{u}_2 = \mathrm{u}_3 = 0$ and $\mathrm{u}_1 = \mathrm{u} = \mathrm{u}(\mathrm{x}, t)$. The only relevant component of $\mathbf{n}$ is $\mathrm{n}_1 = 1$ at x = L and $\mathrm{n}_1 = -1$ at x = 0 and the only non - vanishing strain component is

$$\epsilon := \epsilon_{11} = \epsilon(\mathrm{x}, t) = \mathrm{u}_{,\mathrm{x}} \ . \tag{4.1}$$

For the isotropic KG - Model (cf. Eqs. (3.30) - (3.32), (2.1), (2.2)), the only non - vanishing stress components are

$$\tau := \tau_{11} = E\epsilon = E\mathrm{u}_{,\mathrm{x}} \ , \tag{4.2}$$

$$\mu := \mu_{111} = l^2 E\epsilon_{,\mathrm{x}} = l^2 E\mathrm{u}_{,\mathrm{xx}} \ , \tag{4.3}$$

$$\Sigma := \Sigma_{11} = \tau - \mu_{,\mathrm{x}} = E(\epsilon - l^2\epsilon_{,\mathrm{xx}}) = E(\mathrm{u}_{,\mathrm{x}} - l^2\mathrm{u}_{,\mathrm{xxx}}) \ . \tag{4.4}$$

In the following, repeated use will be made of the dimensionless variables

$$\tilde{\mathrm{x}} := \frac{\mathrm{x}}{L},\ \tilde{\mathrm{u}} := \frac{\mathrm{u}}{L},\ \tilde{l} := \frac{l}{L},\ \tilde{\Sigma} = \frac{\Sigma}{E} \ . \tag{4.5}$$

Omitting body and inertial forces, the field equations (2.14) reduce for the bar problem to

$$\Sigma_{,\mathrm{x}} = 0 \ , \tag{4.6}$$

or equivalently, by using Eq. (4.4),

$$\mathrm{u}_{,\mathrm{xx}} - l^2\mathrm{u}_{,\mathrm{xxxx}} = 0 \ , \tag{4.7}$$

and writing this equation in dimensionless form, we obtain

$$\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}} - \tilde{l}^2\mathrm{u}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}\tilde{\mathrm{x}}\tilde{\mathrm{x}}} = 0 \ . \tag{4.8}$$

Similarly, the boundary conditions (3.38) - (3.41) reduce to

$$\text{either } P_1 = P = \mathrm{n}_1 E(\mathrm{u}_{,\mathrm{x}} - l^2\mathrm{u}_{,\mathrm{xxx}}) \quad \text{or} \quad \mathrm{u} \ , \tag{4.9}$$

$$\text{and either } R_1 = R = l^2 E\mathrm{u}_{,\mathrm{xx}} \quad \text{or} \quad \mathrm{n}_1\mathrm{u}_{,\mathrm{x}} \tag{4.10}$$

have to be prescribed at x = 0 and x = L.

Now assume the bar to be fixed at its left end (x = 0) and that a classical force F_0 acts on its right end (x = L). Thus, the classical boundary conditions (4.9) become

$$\mathrm{u}(0) = 0, \qquad P(L) = E[\mathrm{u}_{,\mathrm{x}} - l^2 \mathrm{u}_{,\mathrm{xxx}}]_{\mathrm{x}=L} = \frac{F_0}{A} \,. \tag{4.11}$$

So far, systematic studies addressing the non-classical boundary conditions do not exist. Usually, the assumption is made that the non - classical forces in Eq. (4.10) are vanishing,

$$R(0) = l^2 E[\mathrm{u}_{,\mathrm{xx}}]_{\mathrm{x}=0} = 0\,, \qquad R(L) = l^2 E[\mathrm{u}_{,\mathrm{xx}}]_{\mathrm{x}=L} = 0\,. \tag{4.12}$$

For reasons of convenience, we introduce the dimensionless traction

$$\tilde{\tau}_0 = \frac{F_0}{EA}\,. \tag{4.13}$$

This, together with the definitions in Eq. (4.5), allows to rewrite equivalently the two boundary conditions (4.11) and (4.12):

$$[\tilde{\mathrm{u}}]_{\tilde{\mathrm{x}}=0} = 0\,, \qquad \left[\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}} - \tilde{l}^2 \mathrm{u}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}\tilde{\mathrm{x}}}\right]_{\tilde{\mathrm{x}}=1} = \tilde{\tau}_0\,, \tag{4.14}$$

$$[\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}}]_{\tilde{\mathrm{x}}=0} = 0\,, \qquad [\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}}]_{\tilde{\mathrm{x}}=1} = 0\,. \tag{4.15}$$

The fourth - order ordinary differential equation (4.8) has the general solution

$$\tilde{\mathrm{u}} = \tilde{a}_1 + \tilde{a}_2 \tilde{\mathrm{x}} + \tilde{a}_3 \mathrm{e}^{\frac{\tilde{\mathrm{x}}}{\tilde{l}}} + \tilde{a}_4 \mathrm{e}^{-\frac{\tilde{\mathrm{x}}}{\tilde{l}}}\,, \tag{4.16}$$

where $\tilde{a}_1$, ..., $\tilde{a}_4$ are constants of integration. By substituting the solution (4.16) into the boundary conditions (4.14), (4.15), we obtain

$$\tilde{a}_1 + \tilde{a}_3 + \tilde{a}_4 = 0\,, \qquad \tilde{a}_2 = \tilde{\tau}_0\,, \qquad \tilde{a}_3 + \tilde{a}_4 = 0\,, \qquad \tilde{a}_3 \sinh\left(\frac{1}{\tilde{l}}\right) = 0\,. \tag{4.17}$$

Provided $\tilde{l} < \infty$, the solution of this system is

$$\tilde{a}_1 = \tilde{a}_3 = \tilde{a}_4 = 0\,, \qquad \tilde{a}_2 = \tilde{\tau}_0\,. \tag{4.18}$$

Consequently, from Eq. (4.16),

$$\tilde{\mathrm{u}} = \tilde{\tau}_0 \tilde{\mathrm{x}} \;\Leftrightarrow\; \epsilon = \tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}} = \tilde{\tau}_0 \quad \text{(classical solution).} \tag{4.19}$$

This is nothing but the solution provided by classical elasticity (classical solution) for the classical boundary conditions $[\tilde{\mathrm{u}}]_{\tilde{\mathrm{x}}=0} = 0$, $[\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}}]_{\tilde{\mathrm{x}}=1} = \tilde{\tau}_0$.

In conclusion, non - classical boundary conditions expressed in terms of vanishing non - classical forces imply homogenous strain distribution (classical solution). Now, it is natural to ask if other non - classical boundary conditions can generate inhomogeneous strain distributions, i.e., non - classical solutions. We shall try to discuss this question in the following.

4.1.2 Effect of non - classical boundary conditions

In all examples, predicted displacement and strain distributions are computed on the basis of solution (4.16), with the constants of integration being determined according to the assumed boundary conditions.

Example 1

Boundary conditions:

$$[\tilde{u}]_{\tilde{x}=0} = 0 , \qquad \left[\tilde{u}_{,\tilde{x}} - \tilde{l}^2 u_{,\tilde{x}\tilde{x}\tilde{x}}\right]_{\tilde{x}=1} = \tilde{\tau}_0 , \tag{4.20}$$

$$-[\tilde{u}_{,\tilde{x}}]_{\tilde{x}=0} = -\, r\, \tilde{\tau}_0 , \qquad [\tilde{u}_{,\tilde{x}}]_{\tilde{x}=1} = r\, \tilde{\tau}_0 . \tag{4.21}$$

with $r \geq 0$ being a proportionality factor. For these boundary conditions, predicted displacement and strain distribution are displayed in Figs. 2, 3.

Remark: It should be remarked, that for some materials widely used, there seems to exist experimental evidence for the so - called gradient stiffening effect. That means, under monotonic loading, the observed material response is stiffer than the one predicted by classical elasticity. Therefore, gradient elasticity is commonly required to predict responses which, under monotonic loading, exhibit the gradient stiffening effect but it is not clear if the gradient stiffening effect should appear for all non - classical boundary conditions or some of them.

For $0 < r < 1$, Fig. 2 shows that a stiffening effect occurs, whereas Fig. 3 shows that no stiffening effect is present if $r > 1$. For $r = 1$, the KG – Model and the classical elasticity predict exactly the same responses.

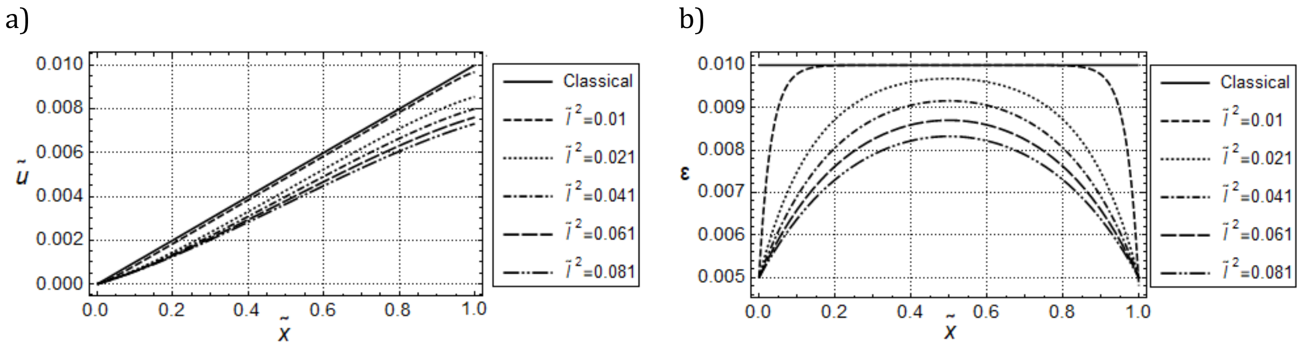


Figure 2: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 0.5$.

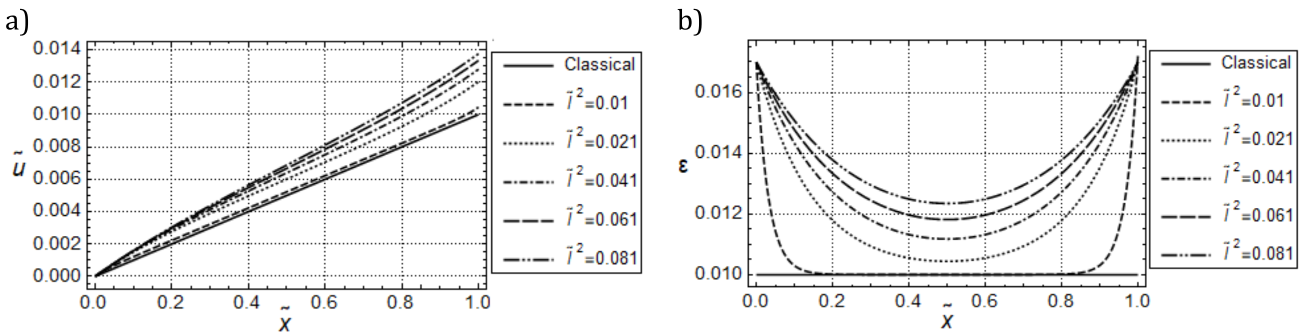


Figure 3: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 1.7$.

For the particular case where $r = 0$, as seen in Fig. 4, displacement and strain distributions show the gradient stiffening effect, but in opposite to case (4.15), the solutions are not classical.

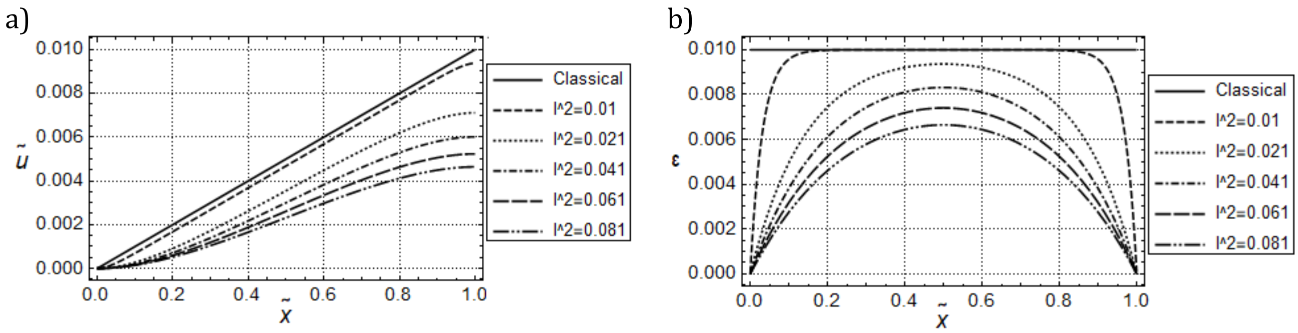


Figure 4: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 0$.

Example 2

Boundary conditions:

$$[\tilde{u}]_{\tilde{x}=0} = 0\,, \qquad [\tilde{u}_{,\tilde{x}} - \tilde{l}^2 u_{,\tilde{x}\tilde{x}\tilde{x}}]_{\tilde{x}=1} = \tilde{\tau}_0\,, \tag{4.22}$$

$$[\tilde{u}_{,\tilde{x}}]_{\tilde{x}=0} = 0\,, \qquad [\tilde{u}_{,\tilde{x}}]_{\tilde{x}=1} = r\,\tilde{\tau}_0\,. \tag{4.23}$$

with $r \geq 0$ being again a proportionality factor. ($r = 0$ is a special case of example 1). Several computed examples suggest to generally take $r < 2$ in order to achieve gradient stiffening effect. Figures 5, 6 illustrate that for $r = 0.8$, gradient stiffening effect occurs (see Fig. 5), whereas for $r = 2.4$, it does not (see Fig. 6). Especially, in the last case, the displacement distributions according to the KG - Model interchange the distribution according to the classical case.

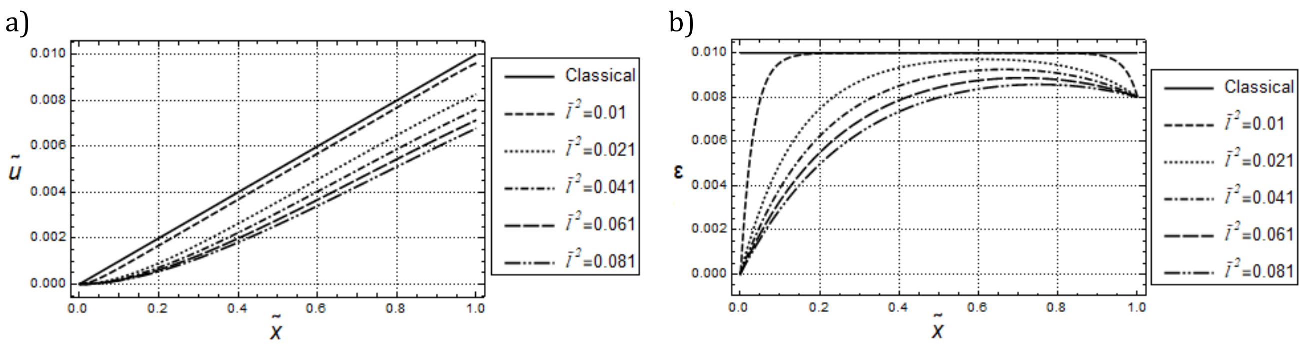


Figure 5: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 0.8$.

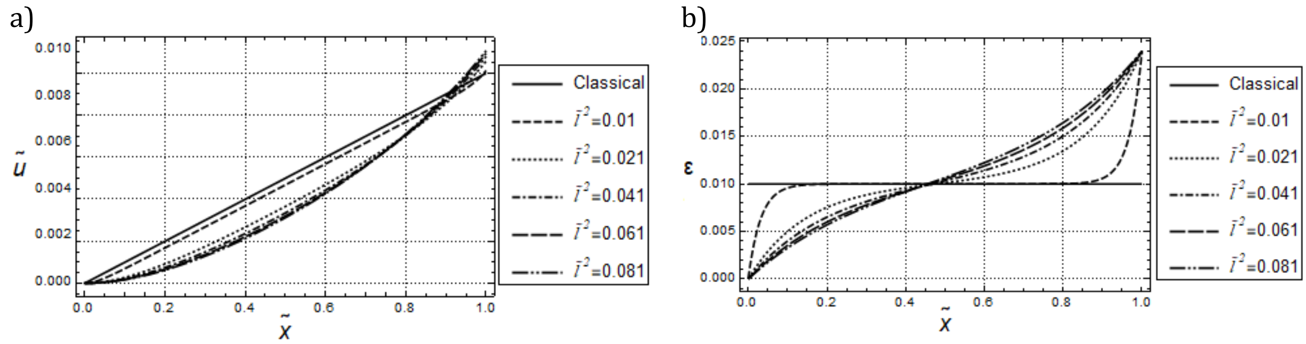


Figure 6: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 2.4$.

Example 3

Boundary conditions:

$$[\tilde{u}]_{\tilde{x}=0} = 0\,, \qquad \left[\tilde{u}_{,\tilde{x}} - \tilde{l}^2 u_{,\tilde{x}\tilde{x}\tilde{x}}\right]_{\tilde{x}=1} = \tilde{\tau}_0\,, \tag{4.24}$$

$$[\tilde{u}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=0} = 0\,, \qquad [\tilde{u}_{,\tilde{x}}]_{\tilde{x}=1} = r\,\tilde{\tau}_0\,. \tag{4.25}$$

where once again $r \geq 0$ is proportionality factor. Figures 7, 8 illustrate that for $r < 1$, the gradient stiffening effect is occurring (see Fig. 7), whereas for $r > 1$ it is not (see Fig. 8).

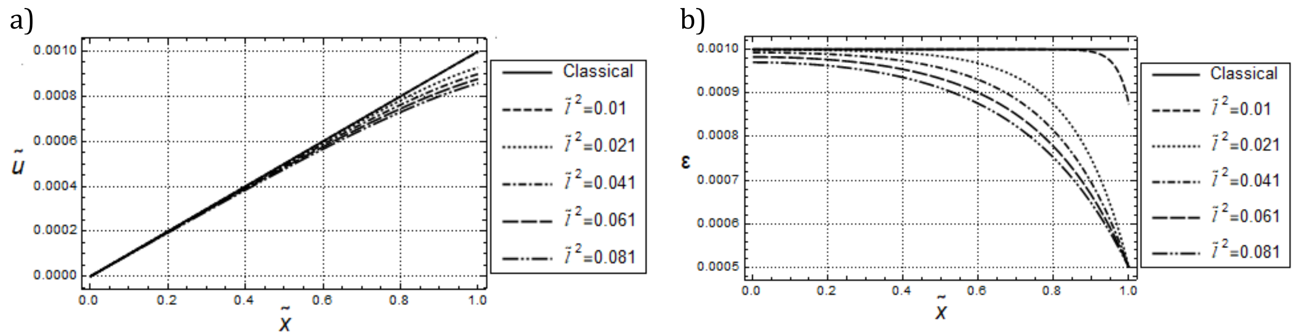


Figure 7: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 0.5$.

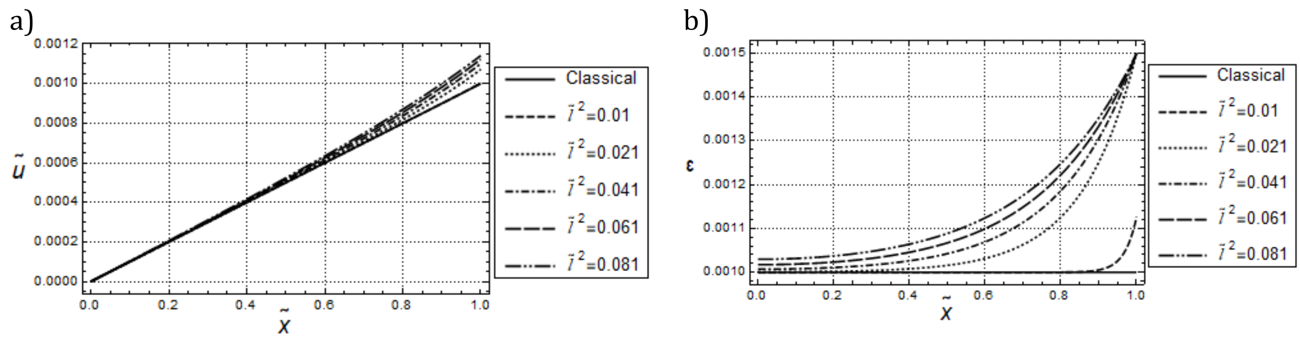


Figure 8: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$, $r = 1.5$.

It should be remarked that somewhat corresponding responses arise whenever the boundary conditions (4.24), (4.25) are replaced by

$$[\tilde{u}]_{\tilde{x}=0} = 0\,, \qquad \left[\tilde{u}_{,\tilde{x}} - \tilde{l}^2 u_{,\tilde{x}\tilde{x}\tilde{x}}\right]_{\tilde{x}=1} = \tilde{\tau}_0\,, \tag{4.26}$$

$$-[\tilde{u}_{,\tilde{x}}]_{\tilde{x}=0} = -\,r\,\tilde{\tau}_0\,, \qquad [\tilde{u}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=1} = 0\,. \tag{4.27}$$

<u>Example 4</u>

Boundary conditions:

$$[\tilde{u}]_{\tilde{x}=0} = 0\,, \qquad \left[\tilde{u}_{,\tilde{x}} - \tilde{l}^2 u_{,\tilde{x}\tilde{x}\tilde{x}}\right]_{\tilde{x}=1} = \tilde{\tau}_0\,, \tag{4.28}$$

$$[\tilde{u}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=0} = r\,\tilde{\tau}_0\,, \qquad [\tilde{u}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=1} = r\,\tilde{\tau}_0\,. \tag{4.29}$$

The non - classical tractions are expressed proportionally to the classical traction $\tilde{\tau}_0$ and are equal at both the left and right end of the beam (*R (L) = R (0)*). For every $r \geq 0$, the predicted displacement and strain distribution are displayed in Fig. 9. The gradient stiffening effect is always occurring.

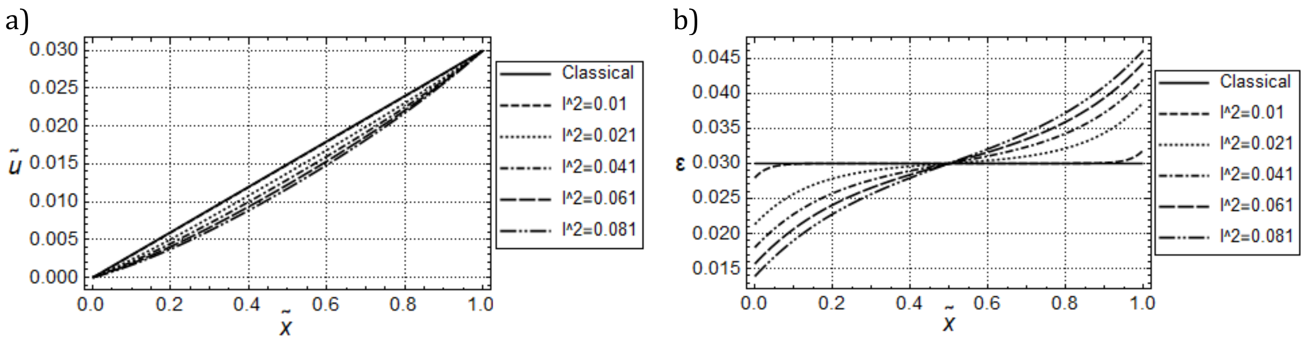


Figure 9: Isotropic KG-Model: Distributions of a) the displacement ũ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}, r \geq 0$.

4.1.3 Discussion

After observing the responses according to all the combinations of boundary conditions calculated above, some conclusions can be drawn.

1. Gradient stiffening effect is always occurring for boundary conditions of the form (4.28) – 4.29). In addition, for the non - classical boundary conditions (4.22) - (4.23), the gradient stiffening effect is partially occurring, and the displacements predicted by the KG - Model interchange the one predicted by classical elasticity. In all other cases of non - classical boundary conditions, depending on the amount of the non - classical boundary conditions, gradient stiffening effect may or may not occur.

2. If both non - classical boundary conditions are of the same type (both of Dirichlet type or both of Neumann type), then strain distributions seem to be symmetrical (see Figs. 2, 3, 4, 9).

3. In all cases, when $\tilde{l} \to 0$, displacement distributions predicted by the KG - Model converge uniformly to the one predicted by classical elasticity, whereas the strain distributions may or may not converge to the classical one.

The calculated examples make clear that the effect of the non - classical boundary conditions is significant both qualitatively and quantitatively. Unfortunately, at the time being, except the case of homogenous non - classical boundary conditions of Neumann type, in all other cases it is not known how to control non - classical boundary conditions. Therefore, and for definiteness, in the remainder of this thesis, we shall focus our attention only on vanishing non - classical tractions, which seems to be a natural hypothesis when no further physical mechanisms are assumed to be active on the boundary.

4.2 Bar loaded by its own weight

Consider the bar of length L in Fig.10, which is loaded by its own weight. The volume force at each position x is constant,

$$F_{11} \equiv F = \rho g \,, \tag{4.30}$$

where ρ is the mass density (see notation in section 2.1) and g is the gravitational acceleration.

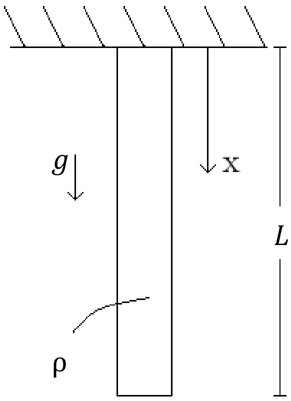


Figure 10: Bar under its own weight.

As in section 4.1, we set the Poisson ration $\nu = 0$ and use the notation $\mathrm{x}_1 = \mathrm{x}$, $\mathrm{u}_2 = \mathrm{u}_3 = 0$, $\mathrm{u}_1 = \mathrm{u} = \mathrm{u(x)}$, $c = \sqrt{E/\rho}$,

$$\epsilon := \epsilon_{11} = \epsilon(\mathrm{x}) = \mathrm{u}_{,\mathrm{x}} \tag{4.31}$$

$$\Sigma := \Sigma_{11} = \Sigma(\mathrm{x}) \tag{4.32}$$

We also use the dimensionless variables and parameters introduced in Eq. (4.5) and in addition we define

$$\tilde{g} := \frac{g}{c^2/L} \,. \tag{4.33}$$

4.2.1 Classical solution

In order to compare the gradient elasticity solution with the classical one, we shall first provide the solution for the classical case. After omitting acceleration terms in Eq.(2.14) and using Eq.(4.30),

$$\Sigma_{,\mathrm{x}} + \rho g = 0 \,. \tag{4.34}$$

By employing Hooke's law $\Sigma = E\epsilon = E\mathrm{u}_{,\mathrm{x}}$,

$$E\mathrm{u}_{,\mathrm{xx}} + \rho g = 0 \;\Rightarrow\; \mathrm{u}_{,\mathrm{xx}} = -\frac{\rho g}{E} = -\frac{g}{c^2} \,, \tag{4.35}$$

or equivalently in dimensionless form,

$$\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}} = -\tilde{g} \,, \tag{4.36}$$

which has the solution

$$\tilde{u} = -\frac{1}{2}\tilde{g}\tilde{x}^2 + \tilde{a}_1\tilde{x} + \tilde{a}_2\,, \tag{4.37}$$

with $\tilde{a}_1$, $\tilde{a}_2$ being constants of integration. Using the boundary conditions

$$[u]_{x=0} = 0\,, \qquad \Sigma(L) = E[u_{,x}]_{x=L} = 0\,, \tag{4.38}$$

or equivalently in dimensionless form,

$$[\tilde{u}]_{\tilde{x}=0} = 0\,, \qquad [\tilde{u}_{,\tilde{x}}]_{\tilde{x}=1} = 0\,, \tag{4.39}$$

we obtain

$$\tilde{a}_1 = \tilde{g}\,, \qquad \tilde{a}_2 = 0\,, \tag{4.40}$$

so that

$$\tilde{u} = -\frac{1}{2}\tilde{g}\tilde{x}^2 + \tilde{g}\tilde{x}\,. \tag{4.41}$$

For the comparative discussions, it is also convenient to refer to the reduced dimensionless displacement $\bar{u}$,

$$\bar{u} = \frac{\tilde{u}}{\tilde{g}} = -\frac{1}{2}\tilde{x}^2 + \tilde{x}\,. \tag{4.42}$$

4.2.2 Responses predicted by the KG - Model

By appealing Eqs. (4.4), (4.30) into Eq. (2.14), we conclude, in the absence of acceleration terms, that

$$E(u_{,xx} - l^2 u_{,xxxx}) = -\rho g, \tag{4.43}$$

or

$$\tilde{u}_{,\tilde{x}\tilde{x}} - \tilde{l}^2 u_{,\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = -\tilde{g}. \tag{4.44}$$

This ordinary differential equation has the general solution

$$\tilde{u} = -\frac{1}{2}\tilde{g}\tilde{x}^2 + \tilde{b}_1 + \tilde{b}_2\tilde{x} + \tilde{b}_3\tilde{l}^2 e^{\tilde{x}/\tilde{l}} + \tilde{b}_4\tilde{l}^2 e^{-\tilde{x}/\tilde{l}}\,, \tag{4.45}$$

with $\tilde{b}_1$, ..., $\tilde{b}_4$ being constants of integration.

We set the classical boundary conditions (cf. Eq. (3.38))

$$u(0) = 0, \qquad P(L) = \Sigma(L) = E[u_{,x} - l^2 u_{,xxx}]_{x=L} = 0\,, \tag{4.46}$$

and the non - classical ones (cf. Eq. (3.39))

$$R(0) = l^2 E[u_{,xx}]_{x=0} = 0, \qquad R(L) = l^2 E[u_{,xx}]_{x=L} = 0\,. \tag{4.47}$$

In dimensionless form,

$$[\tilde{u}]_{\tilde{x}=0} = 0\,, \qquad\qquad [\tilde{u}_{,\tilde{x}} - \tilde{l}^2 \tilde{u}_{,\tilde{x}\tilde{x}\tilde{x}}]_{\tilde{x}=1} = 0\,, \tag{4.48}$$

$$[\tilde{u}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=0} = 0\,, \qquad\qquad [\tilde{u}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=1} = 0\,. \tag{4.49}$$

Using these boundary conditions, we calculate the constants

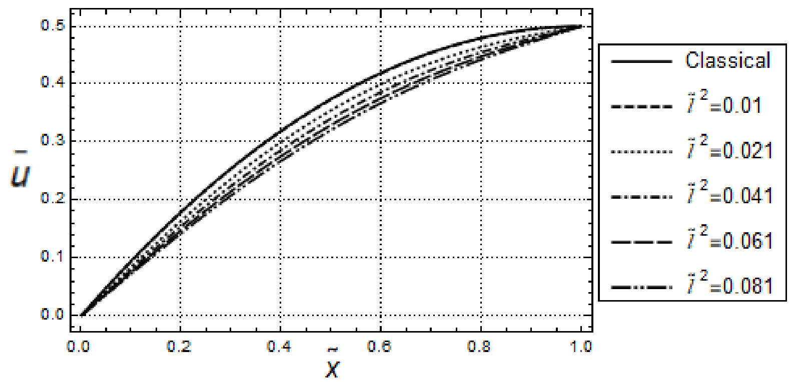
$$\tilde{b}_1 = -\tilde{g}\tilde{l}^2, \qquad \tilde{b}_2 = \tilde{g}, \qquad \tilde{b}_3 = \tilde{g}\frac{1}{1+e^{1/\tilde{l}}}\,, \qquad \tilde{b}_4 = \tilde{g}\frac{e^{1/\tilde{l}}}{1+e^{1/\tilde{l}}}\,, \tag{4.50}$$

and hence,

$$\bar{u} = \frac{\tilde{u}}{\tilde{g}} = -\frac{1}{2}\tilde{x}^2 + \tilde{x} + \frac{\tilde{l}^2}{1+e^{1/\tilde{l}}}e^{\tilde{x}/\tilde{l}} + \frac{\tilde{l}^2 e^{1/\tilde{l}}}{1+e^{1/\tilde{l}}}e^{-\tilde{x}/\tilde{l}} - \tilde{l}^2\,. \tag{4.51}$$

Displacement and strain distributions of the KG - Model and the classical elasticity are presented in Fig. 11. It is clearly seen that the gradient stiffening effect is present. In addition, these distributions converge uniformly to the one predicted by the classical case for $\tilde{l} \to 0$.

a)



b)

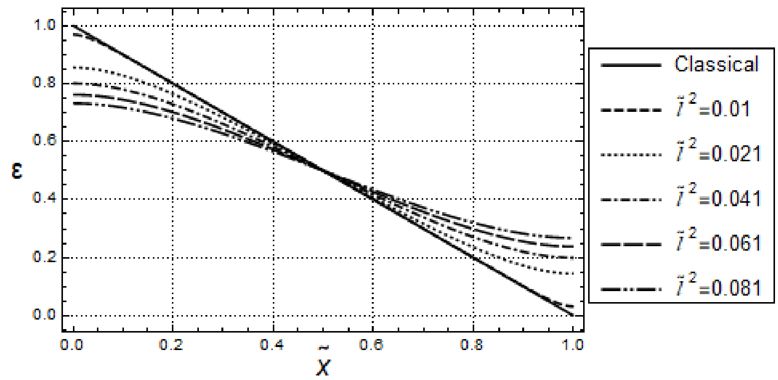


Figure 11: Isotropic KG-Model: Distributions of a) the displacement $\tilde{u}$ and b) the strain $\epsilon = \tilde{u}_{,\tilde{x}}$.

5 Static bending of an Euler - Bernoulli rectangular beam

Most parts of the theory concerning bending in statics and dynamics are inspired by, or even follow, the work on gradient elasticity of Tsakmakis (see Tsakmakis, [34]).

Before going to analyse gradient elastic beams, it is instructive to review the classical Euler - Bernoulli beam theory in a manner convenient for our purposes.

5.1 Review of bending of beams in classical elasticity

5.1.1 The virtual work principle in classical elastostatic

For later reference it is convenient to establish the governing equation of classical elasticity by employing the virtual work principle.

Classical, linear, isotropic (isothermal) elasticity uses for the free energy (called strain energy) the quadratic function

$$\psi = \psi(\epsilon) = \frac{1}{2}\epsilon_{ij}\mathbb{C}_{ijmn}\epsilon_{mn} \,, \tag{5.1}$$

the total energy expended by the internal forces being

$$W^{(i)} \coloneqq \int_V \psi dV \,. \tag{5.2}$$

During a virtual change of displacement $\delta\mathbf{u}$, we have

$$\delta\epsilon_{ij} = \frac{1}{2}\left[\partial_i(\delta \mathrm{u}_j) + \partial_j(\delta \mathrm{u}_i)\right] , \tag{5.3}$$

and

$$\delta W^{(i)} = \int_V \frac{\partial\psi}{\partial\epsilon_{ij}}\delta\epsilon_{ij} dV = \int_V \Sigma_{ij}\delta\epsilon_{ij} dV \,, \tag{5.4}$$

with

$$\Sigma_{ij} = \frac{\partial\psi}{\partial\epsilon_{ij}} = \mathbb{C}_{ijmn}\epsilon_{mn} \,. \tag{5.5}$$

Because of the symmetry of the Cauchy stress $\mathbf{\Sigma}$, we have

$$\begin{aligned} \Sigma_{ij}\delta\epsilon_{ij} &= \frac{1}{2}\left[\Sigma_{ij}\partial_i(\delta \mathrm{u}_j) + \Sigma_{ij}\partial_j(\delta \mathrm{u}_i)\right] = \frac{1}{2}\left[\Sigma_{ij}\partial_i(\delta \mathrm{u}_j) + \Sigma_{ji}\partial_i(\delta \mathrm{u}_j)\right] \\ &= \Sigma_{ij}\partial_i(\delta \mathrm{u}_j) = \partial_i(\Sigma_{ij}\delta \mathrm{u}_j) - (\partial_i\Sigma_{ij})\delta \mathrm{u}_j \,, \end{aligned} \tag{5.6}$$

so that, from Eq. (5.4),

$$\delta W^{(i)} = \int_V \partial_i(\Sigma_{ij}\delta \mathrm{u}_j)dV - \int_V (\partial_i\Sigma_{ij})\delta \mathrm{u}_j dV\,. \tag{5.7}$$

The first integral on the right hand side can be recast with the aid of the divergence theorem, to obtain

$$\delta W^{(i)} = \int_{\partial V} \mathrm{n}_i\Sigma_{ij}\delta \mathrm{u}_j dS - \int_V (\partial_i\Sigma_{ij})\delta \mathrm{u}_j dV\,. \tag{5.8}$$

Omitting inertial effects, the virtual work principle requires that the virtual work expended by the external forces, $\delta W^{(e)}$, is equal to the virtual work expended by the internal forces, $\delta W^{(i)}$,

$$\delta W^{(e)} = \delta W^{(i)}\,. \tag{5.9}$$

The form of $\delta W^{(i)}$ in Eq. (5.8) suggests assuming the existence of contact tractions **t** and body forces **F** and setting

$$\delta W^{(e)} := \int_{\partial V} t_j\delta \mathrm{u}_j dS + \int_V F_j\delta \mathrm{u}_j dV\,. \tag{5.10}$$

After some rearrangement of terms, it follows from Eqs. (5.8) - (5.10) that

$$\int_{\partial V} (t_j - \mathrm{n}_i\Sigma_{ij})\delta \mathrm{u}_j dS + \int_V (\partial_i\Sigma_{ij} + F_j)\delta \mathrm{u}_j dV = 0. \tag{5.11}$$

As usual, the functions $\delta \mathrm{u}_i$ are vanishing on the part of ∂V where displacement boundary conditions are prescribed and otherwise, they are arbitrary continuous functions on both the part of ∂V where the tractions t_i are prescribed and on V. Thus, in view of the arbitrariness of the functions $\delta \mathrm{u}_i$, Eq. (5.11) implies the equilibrium equations

$$\partial_i\Sigma_{ij} + F_j = 0\,, \tag{5.12}$$

and the boundary conditions

$$\text{either } t_j = \mathrm{n}_i\Sigma_{ij} \text{ or } \mathrm{u}_j \tag{5.13}$$

have to be prescribed on ∂V.

5.1.2 Euler - Bernoulli beam theory - The engineering mechanics approach

A) Bending Geometry

For the rectangular beam in Fig. 12 we use a Cartesian coordinate system $\{\mathrm{x}, \mathrm{y}, \mathrm{z}\}$ with the origin 0 being located on the left boundary plane. The z - axis is a symmetry axis and the x - axis is the neutral axis, the latter being defined to experience no change in length. Component representations will refer to this coordinate system. The beam has length L, width 2b, height 2c, constant cross - section A(x) = A and is subject to a transverse distributed static load per unit length q(x) along the x - axis. In addition, problem specific reaction forces or external forces are assumed to apply at x = 0, L. For the components of **u** and **Σ** we assume that

$$\mathbf{u} = \mathbf{u}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{u} \\ 0 \\ \mathrm{w} \end{pmatrix}, \tag{5.14}$$

$$\mathbf{\Sigma} = \mathbf{\Sigma}(\mathrm{x}, \mathrm{z}) \triangleq \begin{pmatrix} \sigma_x & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & \sigma_z \end{pmatrix} . \tag{5.15}$$

This corresponds to plane stress state. Omitting body forces, the equilibrium equations (5.12) become

$$\frac{\partial \sigma_x}{\partial \mathrm{x}} + \frac{\partial \tau}{\partial \mathrm{z}} = 0 \, , \tag{5.16}$$

$$\frac{\partial \tau}{\partial \mathrm{x}} + \frac{\partial \sigma_z}{\partial \mathrm{z}} = 0 \, . \tag{5.17}$$

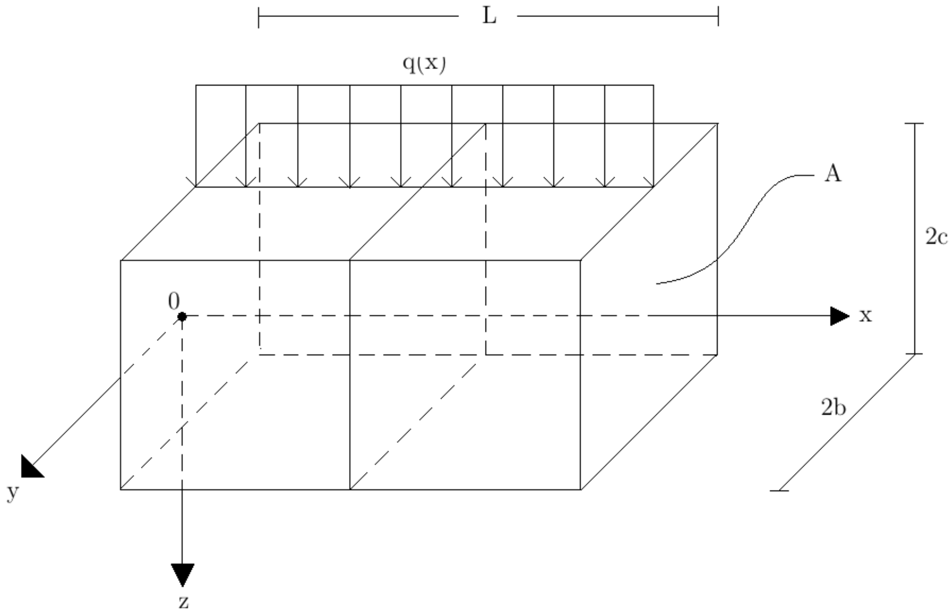


Figure 12: Rectangular beam of length L, width 2b, height 2c and constant cross - section A subject to an external transverse distributed load q(x) per unit length along the x - axis, and to problem specific reaction forces or external load at x = 0, L.

The fundamental assumption (hypothesis) of the Euler - Bernoulli beam theory is that the plane cross - sections A remain plane and perpendicular to the deformed x - axis and that the shape of the cross - sections does not change (see Bauchau and Craig [5], Section 5.1 and 5.4.2). This means that the cross - sections undergo rigid body motions (no deformation). Since the deformations are supposed to be small (see Section 2.1), the assumption of inextensibility in the z - direction implies vanishing $\epsilon_{33} = \epsilon_z$ strain,

$$\epsilon_z = \frac{\partial \mathrm{w}(\mathrm{x},\mathrm{z})}{\partial \mathrm{z}} = 0 \, . \tag{5.18}$$

This in turn implies that the deflection w does not depend on z,

$$\mathrm{w} = \mathrm{w}(\mathrm{x}) \, . \tag{5.19}$$

As usual, we set w = w(x) to be the displacement component of the neutral axis. For the beam in Fig. 12 the Euler - Bernoulli hypothesis imposes further on $\epsilon_{11} = \epsilon_x$ the form (see Fig. 13),

$$\epsilon_x = \epsilon_x(\mathrm{x},\mathrm{z}) = \frac{\Delta s - \Delta s_0}{\Delta s_0} = \frac{(\rho + \mathrm{z})\Delta\Phi - \rho\Delta\Phi}{\rho\Delta\Phi} \tag{5.20}$$

or

$$\epsilon_x = \frac{1}{\rho} z \, . \tag{5.21}$$

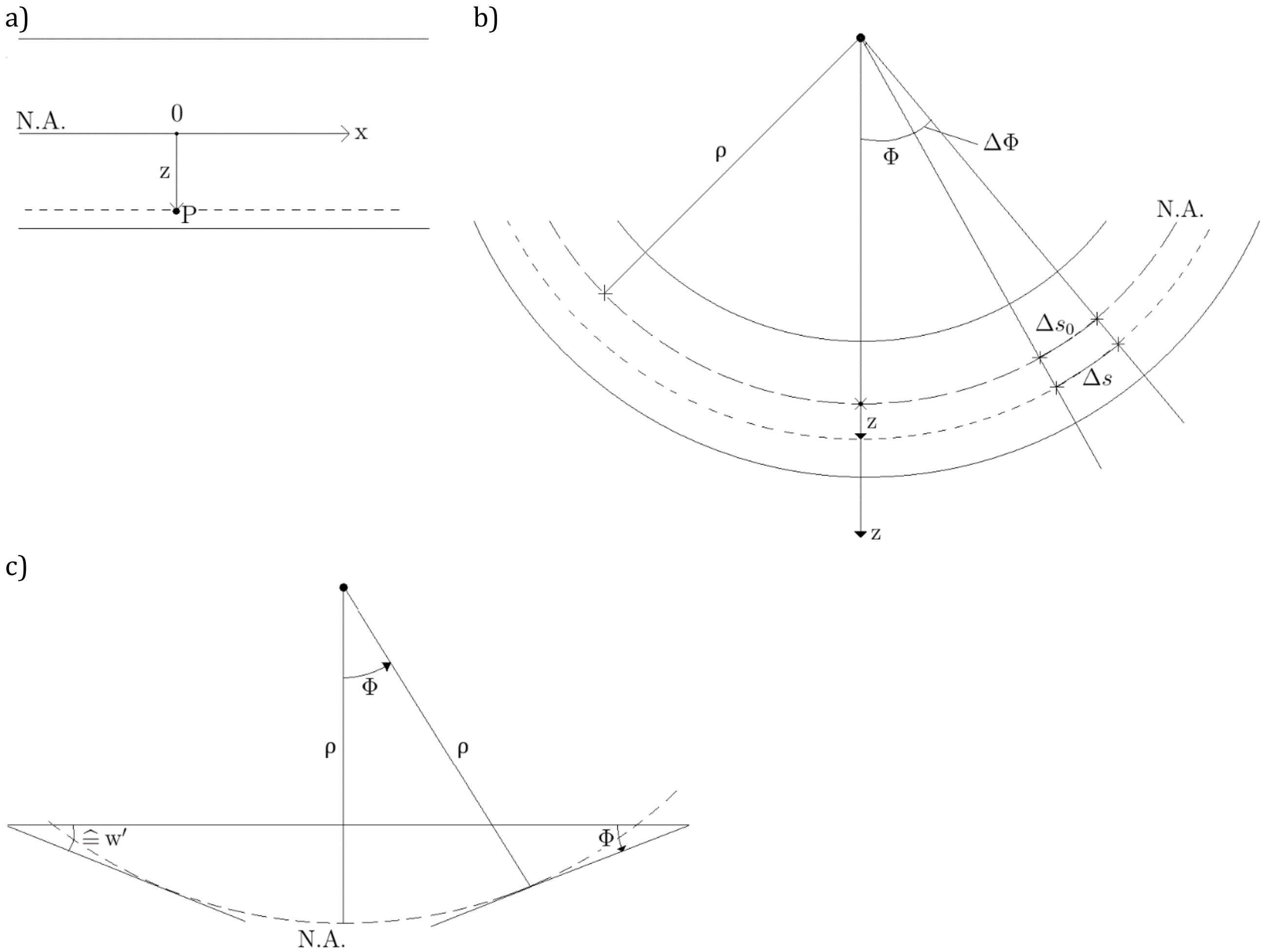


Figure 13: Bending geometry of the Euler - Bernoulli beam: s = s (x, z) denotes the arc length assigned to some point P = (x, z), $s_0 = s_0(x)$ is the arc length assigned to the point on the neutral axis with the same x as for the point P and $\rho = \rho(x)$ is the curvature radius (not to be confused with the mass density in previous sections). Further, for small deformations, $\Phi \approx \tan\Phi = -w'$.

In terms of deflection w = w(x),

$$\frac{1}{\rho} = \Phi' = -w'' \, , \tag{5.22}$$

where

$$(\,)' = \frac{d}{dx} \, . \tag{5.23}$$

Thus, the strain ϵ_x becomes

$$\epsilon_x = \frac{\partial u}{\partial x} = -w''z\,, \tag{5.24}$$

and after integration, we can deduce for the Euler - Bernoulli beam, that

$$u = -w'z\,, \tag{5.25}$$

and hence

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \mathrel{\widehat{=}} \begin{pmatrix} -w'z \\ 0 \\ w \end{pmatrix}. \tag{5.26}$$

An important consequence of the Euler - Bernoulli beam geometry is that, besides the strain ϵ_y , the shear strain $\epsilon_{13} = \epsilon_{xz}$ vanishes as well:

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = -w' + w' = 0\,. \tag{5.27}$$

It follows that the whole strain tensor has only one component,

$$\boldsymbol{\epsilon} \mathrel{\widehat{=}} \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.28}$$

B) Stresses – Section forces

It is outlined that an isotropic elasticity law (cf. Eqs. (5.5), (2.1)) is chosen commonly in engineering mechanics, so that the stress component σ_x becomes

$$\sigma_x = E\epsilon_x\,, \tag{5.29}$$

and therefore, in view of Eq. (5.24),

$$\sigma_x = -Ew''z\,. \tag{5.30}$$

As before, E denotes the Young's Modulus. Now, it is well known that, for arbitrary values of Poisson's ratio $0 \le \nu \le \frac{1}{2}$, the assumed strain state (5.28) is not consistent with the stress state (5.15). In particular, the strain state (5.28) and the elasticity law imply vanishing shear stress $\boldsymbol{\tau}$ always, which is not true for general loading conditions as, e.g., in Fig. 12. However, this is ignored in engineering mechanics and a shear stress $\boldsymbol{\tau}$ is assumed to exist. This inconsistency is accepted by interpreting it to be negligible (see, e.g., [21]). In addition, the assumed Euler - Bernoulli kinematics leads to field equations, which cannot be satisfied in local form and/or not all boundary conditions in local form can be satisfied. To overcome this difficulty, one refrains from fulfilling the equilibrium equations and the boundary conditions pointwise. One is satisfied with the fulfilment of these equations generally in global form, i.e., in the form of averages expressed by integrals over cross - sections, which represent section "forces". For the present case, the section "forces" are the normal force, the shearing force, and the bending moment. These forces can be introduced as follows.

Let $\Delta\mathfrak{B}$ be a part of the beam in Fig. 12, which is bounded on the right side by a section plane A with outward normal $\mathbf{n} = \mathbf{e}_x$ and on the left side by a section plane A with outward normal $\mathbf{n} = -\,\mathbf{e}_x$. The traction vectors on these planes and the virtual displacement vector are given by (cf. Eqs. (5.13), (5.26)),

$$\mathbf{t} = \mathbf{t}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} t_1 \\ 0 \\ t_3 \end{pmatrix} = \begin{pmatrix} \mathrm{n}_1 \Sigma_{11} \\ 0 \\ \mathrm{n}_1 \Sigma_{13} \end{pmatrix}, \tag{5.31}$$

$$\delta\mathbf{u} = \delta\mathbf{u}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} -\delta \mathrm{w}'\mathrm{z} \\ 0 \\ \delta \mathrm{w} \end{pmatrix}, \tag{5.32}$$

while the virtual work (more precisely the increment of virtual work) expended by **t** on each of the section planes A is

$$\int_A \mathbf{t} \cdot \delta\mathbf{u}\, dS = \int_A \begin{pmatrix} t_1 \\ 0 \\ t_3 \end{pmatrix} \cdot \begin{pmatrix} -\delta \mathrm{w}'\mathrm{z} \\ 0 \\ \delta \mathrm{w} \end{pmatrix} dS = \left(\int_A t_1 \mathrm{z} dS\right)(-\delta \mathrm{w}') + \left(\int_A t_3 dS\right)\delta \mathrm{w}$$

$$= \left(\int_A \sigma_\mathrm{x} \mathrm{z} dS\right) \mathrm{n}_1 (-\delta \mathrm{w}') + \left(\int_A \tau dS\right) \mathrm{n}_1 \delta \mathrm{w}\,. \tag{5.33}$$

In order to rewrite this equation, we define section displacement vector **f**, section rotation vector $\boldsymbol{\phi}$, section shearing force **Q** and section bending moment **M**, by

$$\mathbf{f} = \mathbf{f}(\mathrm{x}) = \mathrm{w}\mathbf{e}_\mathrm{z}\,, \tag{5.34}$$

$$\boldsymbol{\phi} = \boldsymbol{\phi}(\mathrm{x}) = (-\mathrm{w}')\mathbf{e}_\mathrm{y}\,, \tag{5.35}$$

$$\mathbf{Q} = \mathbf{Q}(\mathbf{n}, \mathrm{x}) = Q\mathrm{n}_1\mathbf{e}_\mathrm{z}\,, \tag{5.36}$$

$$\mathbf{M} = \mathbf{M}(\mathbf{n}, \mathrm{x}) = M\mathrm{n}_1\mathbf{e}_\mathrm{y}\,, \tag{5.37}$$

with

$$Q = Q(\mathrm{x}) = \int_A \tau\, dS\,, \tag{5.38}$$

$$M = M(\mathrm{x}) = \int_A \sigma_\mathrm{x}\, z\, dS\,, \tag{5.39}$$

being scalar (section bending) moment and scalar (section shearing) force, respectively. Then, the virtual work expended by **t** on A becomes

$$\int_A \mathbf{t} \cdot \delta\mathbf{u} dS = \mathbf{M} \cdot \delta\boldsymbol{\phi} + \mathbf{Q} \cdot \delta\mathbf{f} = M(\mathrm{x})\mathrm{n}_1\big(-\delta \mathrm{w}'(\mathrm{x})\big) + Q(\mathrm{x})\mathrm{n}_1 \delta \mathrm{w}(\mathrm{x})\,. \tag{5.40}$$

The section forces **Q**, **M** are work conjugate to **f** and $\boldsymbol{\phi}$, respectively, and since $\delta \mathrm{w}$ and $\delta \mathrm{w}'$ can be chosen arbitrarily on the boundary planes A, they are statically equivalent to the force components induced by t. Furthermore, we define (see Appendix A) the normal section force vector **N** by

$$\mathbf{N} = \mathbf{N}(\mathbf{n}, \mathrm{x}) = \left(\int_A t_1 dS\right)\mathbf{e}_\mathrm{x} = \int_A \mathrm{n}_1 \Sigma_{11} dS\ \mathbf{e}_\mathrm{x} = N\mathrm{n}_1 \mathrm{e}_\mathrm{x} \equiv N\mathbf{n}\,, \tag{5.41}$$

with N being the scalar normal force

$$N = N(\mathrm{x}) = \int_A \sigma_\mathrm{x}\, dS\,. \tag{5.42}$$

In the following, we shall pursue the tradition of engineering mechanics, so we shall formulate the theory in terms of *Q, M* and *N*.

C) Boundary conditions

According to Fig. 14, the boundary surface ∂V of the beam is the union of the boundary planes $\mathrm{A}^{①}, \dots, \mathrm{A}^{⑥}$. On every $\mathrm{A}^{©}$, C= 1, …, 6, the relations

$$\mathbf{t} = \mathbf{t}(\mathrm{x},\mathrm{z}) \,\widehat{=} \begin{pmatrix} t_1 \\ 0 \\ t_3 \end{pmatrix}, \qquad \mathbf{n} \,\widehat{=} \begin{pmatrix} \mathrm{n}_1 \\ \mathrm{n}_2 \\ \mathrm{n}_3 \end{pmatrix}, \tag{5.43}$$

$$t_j = \mathrm{n}_i \Sigma_{ij} \Rightarrow \begin{pmatrix} t_1 \\ 0 \\ t_3 \end{pmatrix} = \begin{pmatrix} \mathrm{n}_1\sigma_\mathrm{x} + \mathrm{n}_3\tau \\ 0 \\ \mathrm{n}_1\tau + \mathrm{n}_3\sigma_\mathrm{z} \end{pmatrix}, \tag{5.44}$$

hold. In detail, we set the following boundary conditions.

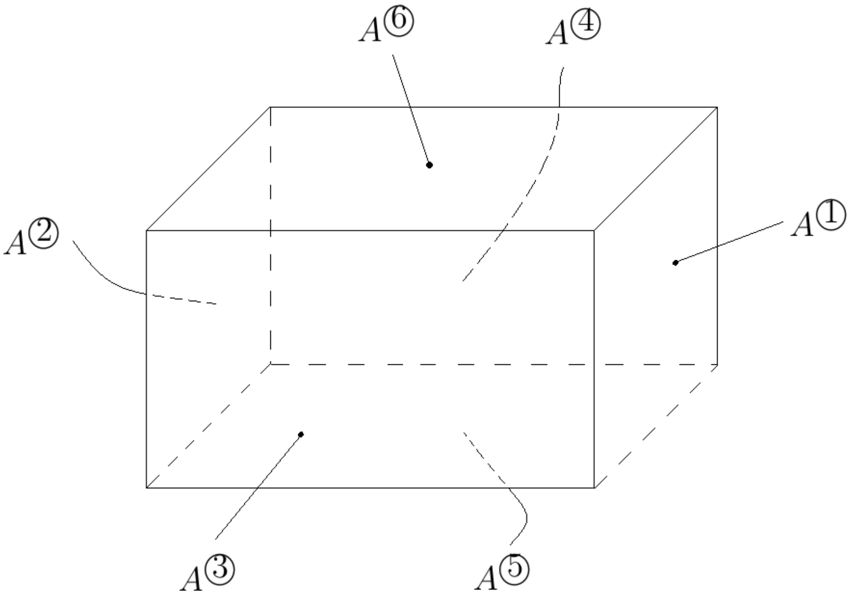


Figure 14: The boundary planes of the beam in Fig. 12.

Plane $\mathrm{A}^{①}$

$$\mathrm{x} = L\,, \qquad \mathbf{n} = \mathbf{e}_\mathrm{x} \,\widehat{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.45}$$

The forces *Q(L), M(L)* are statically equivalent to the forces induced by **t** on $\mathrm{A}^{①}$ and work conjugate to $\delta\mathrm{w}(L)$, $\delta\mathrm{w}'(L)$ respectively. On this plane we set the global boundary conditions

$$\text{either } Q_L = Q(L) \text{ or } \mathrm{w}(L) \text{ and} \tag{5.46}$$

$$\text{either } M_L = M(L) \text{ or } \mathrm{w}'(L) \tag{5.47}$$

have to be prescribed. Further, we assume

$$N_L = N(L) = 0\,. \tag{5.48}$$

<u>Plane A②</u>

$$\mathrm{x} = 0\,, \qquad \mathbf{n} = -\mathbf{e}_\mathrm{x} \mathrel{\widehat{=}} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.49}$$

Similar to plane A①, we set the global boundary conditions

$$\text{either } Q_0 = Q(0) \text{ or } \mathrm{w}(0) \text{ and} \tag{5.50}$$

$$\text{either } M_0 = M(0) \text{ or } \mathrm{w}'(0) \tag{5.51}$$

have to be prescribed on A②.

<u>Planes A③, A④</u>

$$\mathrm{y} = \pm b\,, \qquad \mathbf{n} = \pm\mathbf{e}_\mathrm{y}\,. \tag{5.52}$$

These lateral planes are supposed to be traction - free, corresponding to homogenous traction boundary conditions in local and global form.

<u>Planes A⑤, A⑥</u>

$$\mathrm{z} = \pm c\,, \qquad \mathbf{n} = \pm\mathbf{e}_\mathrm{z} \mathrel{\widehat{=}} \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}. \tag{5.53}$$

We set the local conditions (cf. Eq. (5.44))

$$\begin{pmatrix} t_1 \\ 0 \\ t_3 \end{pmatrix}_{z=c} = \begin{pmatrix} \tau \\ 0 \\ \sigma_\mathrm{z} \end{pmatrix}_{z=c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.54}$$

$$\begin{pmatrix} t_1 \\ 0 \\ t_3 \end{pmatrix}_{z=-c} = \begin{pmatrix} -\tau \\ 0 \\ -\sigma_\mathrm{z} \end{pmatrix}_{z=-c} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{q(x)}{2b} \end{pmatrix}. \tag{5.55}$$

D) Governing equations – centroidal axis

In engineering mechanics, the global equilibrium equations for the problem in Fig. 12 are derived from Eqs. (5.16), (5.17) as follows. Integration of Eq. (5.16) over A yields

$$\int_A \frac{\partial \sigma_\mathrm{x}}{\partial \mathrm{x}}\, dS + \int_A \frac{\partial \tau}{\partial \mathrm{z}}\, dS = 0\,, \tag{5.56}$$

or

$$\frac{d}{dx}\int_A \sigma_x dS + \int_{y=-b}^{y=b}\int_{z=-c}^{z=c}\frac{\partial\tau}{\partial z}dz\,dy = 0\,. \tag{5.57}$$

Therefore

$$\frac{d}{dx}\int_A \sigma_x dS + 2b[\tau]_{z=-c}^{z=c} = 0\,, \tag{5.58}$$

and in view of Eqs. (5.54), (5.55), (5.41), (5.48),

$$\frac{d}{dx}N = \frac{d}{dx}\int_A \sigma_x dS = 0 \;\Rightarrow\; N = \int_A \sigma_x dS = 0\,. \tag{5.59}$$

From Eq. (5.30), $\sigma_x = E\epsilon_x = -E\mathrm{w}''z$, so that Eq. (5.59) implies

$$-E\mathrm{w}''\int_A z\,dS = 0\,, \tag{5.60}$$

and since w'' does not vanish identically,

$$\int_A z\,dS = 0\,. \tag{5.61}$$

Keeping in mind that the x - axis is the neutral axis, Eq. (5.61) indicates that the x - axis is also a centroidal axis.

Next, integration of the local equilibrium equation (5.17) over A, at arbitrary x, yields

$$\int_A \frac{\partial\tau}{\partial x}dS + \int_A \frac{\partial\sigma_z}{\partial z}dS = 0\,. \tag{5.62}$$

Using similar steps as above,

$$\frac{d}{dx}\int_A \tau dS + 2b[\sigma_z]_{z=-c}^{z=c} = 0\,, \tag{5.63}$$

and by taking into account Eqs. (5.38), (5.54), (5.55),

$$Q' + q = 0\,. \tag{5.64}$$

The two Eqs. (5.59), (5.64) are the global forms of the equilibrium equations for forces. In order to obtain the corresponding equation for moments, we multiply Eq. (5.16) by z and integrate the result over A, at arbitrary position x:

$$\int_A \frac{\partial\sigma_x}{\partial x}z\,dS + \int_A \frac{\partial\tau}{\partial z}z\,dS = 0\,. \tag{5.65}$$

Applying partial integration in the second integral,

$$\frac{d}{d\mathrm{x}}\int_A \sigma_{\mathrm{x}} \mathrm{z} dS + \int_{\mathrm{y}=-b}^{\mathrm{y}=b}\int_{\mathrm{z}=-c}^{\mathrm{z}=c} \frac{\partial}{\partial \mathrm{z}}(\tau \mathrm{z})\, d\mathrm{z}\, d\mathrm{y} - \int_A \tau dS = 0\,, \tag{5.66}$$

or

$$M' + 2b[\tau \mathrm{z}]_{\mathrm{z}=-c}^{\mathrm{z}=c} - Q = 0\,, \tag{5.67}$$

where use has been made of Eqs. (5.38), (5.39). By invoking the boundary conditions (5.54), (5.55) once again, we obtain the global equilibrium equation for moments

$$M' - Q = 0\,. \tag{5.68}$$

Once more, the aim of engineering mechanics is to express the theory in terms of the global variables N, Q, M, which are related to each other and to the external load q by Eqs. (5.59), (5.64), (5.68). In order also to relate these variables to the kinematic of the bending, we turn back to the elasticity law (5.30) and substitute it in the relation (5.39):

$$M = -E\mathrm{w}'' \int_A \mathrm{z}^2 dS\,. \tag{5.69}$$

Thus, we obtain the well - known relation

$$M = -EI\mathrm{w}''\,, \tag{5.70}$$

with $I \equiv I_{\mathrm{y}}$ denoting the moment of inertia about the y - axis,

$$I := \int_A \mathrm{z}^2 dS\,. \tag{5.71}$$

The term EI is called **flexural rigidity**.

Altogether, Eqs. (5.64), (5.68) and the global elasticity law (5.70), together with condition (5.59), are the governing equations for the Euler - Bernoulli beam theory, which may be recast also in the form

$$M'' + q = 0\,, \tag{5.72}$$

or

$$EI\mathrm{w}'''' = q\,. \tag{5.73}$$

Last equation is known as the **deflection curve**. The boundary conditions for solving the differential equation (5.73) are given in Eqs. (5.46), (5.47), (5.50), (5.51). They can be expressed in terms of flexure w by invoking to Eqs. (5.68), (5.70),

$$M = -EI\mathrm{w}''\,, \qquad Q = -EI\mathrm{w}'''\,, \tag{5.74}$$

leading to

$$\text{either } Q_L = -EI\mathrm{w}'''(L) \text{ or } \mathrm{w}(L), \tag{5.75}$$

$$\text{either } M_L = -EI\mathrm{w}''(L) \text{ or } \mathrm{w}'(L), \tag{5.76}$$

$$\text{either } Q_0 = -EI\mathrm{w}'''(0) \text{ or } \mathrm{w}(0), \tag{5.77}$$

$$\text{either } M_0 = -EI\mathrm{w}''(0) \text{ or } \mathrm{w}'(0), \tag{5.78}$$

have to be prescribed.

E) One – dimensional virtual work principle

For elastic materials, the work expended by external forces will be stored in the material as energy expended by the internal forces, the latter being given in Eq. (5.2). This is an energy balance law, which can be expressed in the form of the virtual work principle (5.9). From Eqs. (5.1), (2.1), (2.2), (5.28),

$$\psi = \frac{1}{2} E \epsilon_{\mathrm{x}}^2 \,, \tag{5.79}$$

so that $W^{(i)}$ in Eq. (5.2) becomes

$$W^{(i)} = \frac{1}{2} \int_V E \, \epsilon_{\mathrm{x}}^2 dV \,, \tag{5.80}$$

which is a functional of the strain ϵ_{x} . On the other hand, $\epsilon_{\mathrm{x}} = -\mathrm{w}''\mathrm{z}$ for the Euler - Bernoulli beam, and therefore

$$W^{(i)} = \frac{1}{2} \int_V E \, (\mathrm{w}'')^2 \mathrm{z}^2 dV = \frac{1}{2} \int_{\mathrm{x}=0}^{\mathrm{x}=L} E \, (\mathrm{w}'')^2 \left(\int_A \mathrm{z}^2 dS \right) d\mathrm{x} \,, \tag{5.81}$$

or equivalently, by virtue of Eq. (5.71),

$$W^{(i)} = \frac{1}{2} \int_0^L EI \, (\mathrm{w}'')^2 d\mathrm{x} \,, \tag{5.82}$$

which is a one - dimensional functional of the flexure w. It follows for the variation $\delta W^{(i)}$, that

$$\delta W^{(i)} = EI \int_0^L \mathrm{w}''(\mathrm{x}) \delta \mathrm{w}''(\mathrm{x}) \, d\mathrm{x} \,. \tag{5.83}$$

Applying partial integration,

$$\begin{aligned} \delta W^{(i)} &= EI \int_0^L \big(\mathrm{w}''(\mathrm{x}) \, \delta \mathrm{w}'(\mathrm{x})\big)' d\mathrm{x} - EI \int_0^L \mathrm{w}'''(\mathrm{x}) \, \delta \mathrm{w}'(\mathrm{x}) \, d\mathrm{x} \\ &= EI [\mathrm{w}''(\mathrm{x}) \, \delta \mathrm{w}'(\mathrm{x})]_{\mathrm{x}=0}^{\mathrm{x}=L} - EI \int_0^L \mathrm{w}'''(\mathrm{x}) \, \delta \mathrm{w}'(\mathrm{x}) \, d\mathrm{x} \,. \end{aligned} \tag{5.84}$$

Integrating partially once more,

$$\delta W^{(i)} = -EI \big[\mathrm{w}''(\mathrm{x}) \big(-\delta \mathrm{w}'(\mathrm{x})\big)\big]_{\mathrm{x}=0}^{\mathrm{x}=L} - EI [\mathrm{w}'''(\mathrm{x}) \, \delta \mathrm{w}(\mathrm{x})]_{\mathrm{x}=0}^{\mathrm{x}=L} + EI \int_0^L \mathrm{w}''''(\mathrm{x}) \, \delta \mathrm{w}(\mathrm{x}) \, d\mathrm{x} \,. \tag{5.85}$$

When setting up the virtual work $\delta W^{(e)}$ we have to take into account that the boundary planes A③, A④ and A⑤ are free of externally acting forces (see Figs. 12, 14). Hence,

$$\delta W^{(e)} = \int_{\mathrm{A}^{①}\cup\mathrm{A}^{②}\cup\mathrm{A}^{③}} \mathbf{t}\cdot\delta\mathbf{u}\,dS\,. \tag{5.86}$$

Keeping in mind that on the section planes A①, A② the boundary conditions are imposed globally (cf. Eqs. (5.46), (5.47) and (5.50), (5.51)), and that on such planes Eq. (5.40) applies, we can establish the following results.

$$\mathrm{x} = L, \mathrm{n}_1 = 1: \int_{\mathrm{A}^{①}} \mathbf{t}\cdot\delta\mathbf{u}\,dS = Q(L)\delta\mathrm{w}(L) + M(L)\big(-\delta\mathrm{w}'(L)\big)\,, \tag{5.87}$$

$$\mathrm{x} = 0, \mathrm{n}_1 = -1: \int_{\mathrm{A}^{②}} \mathbf{t}\cdot\delta\mathbf{u}\,dS = -\big[Q(0)\delta\mathrm{w}(0) + M(0)\big(-\delta\mathrm{w}'(0)\big)\big]\,. \tag{5.88}$$

On the boundary plane A⑥, the boundary conditions are imposed locally (cf. Eq. (5.55)), so that, in view of Eq. (5.32),

$$\int_{\mathrm{A}^{⑥}} \mathbf{t}\cdot\delta\mathbf{u} dS = \int_{\mathrm{A}^{⑥}} (t_3\mathbf{e}_z)\cdot(\delta\mathrm{w}\mathbf{e}_z) dS = \int_{\mathrm{x}=0}^{\mathrm{x}=L}\int_{\mathrm{y}=-b}^{\mathrm{y}=b} \frac{q(\mathrm{x})}{2b}\delta\mathrm{w}(\mathrm{x}) dy\, dx$$

$$= \int_{\mathrm{x}=0}^{\mathrm{x}=L} q(\mathrm{x})\delta\mathrm{w}(\mathrm{x}) d\mathrm{x}\,. \tag{5.89}$$

Altogether,

$$\delta W^{(e)} = [Q(\mathrm{x})\delta\mathrm{w}(\mathrm{x})]_{\mathrm{x}=0}^{\mathrm{x}=L} + \big[M(\mathrm{x})\big(-\delta\mathrm{w}'(\mathrm{x})\big)\big]_{\mathrm{x}=0}^{\mathrm{x}=L} + \int_{\mathrm{x}=0}^{\mathrm{x}=L} q(\mathrm{x})\delta\mathrm{w}(\mathrm{x}) dx\,. \tag{5.90}$$

On substituting Eqs. (5.85) and (5.90) into Eq. (5.9),

$$\big[\big(Q(\mathrm{x}) + EI\mathrm{w}'''(\mathrm{x})\big)\delta\mathrm{w}(\mathrm{x})\big]_{\mathrm{x}=0}^{\mathrm{x}=L} + \big[\big(M(\mathrm{x}) + EI\mathrm{w}''(\mathrm{x})\big)\big(-\delta\mathrm{w}'(\mathrm{x})\big)\big]_{\mathrm{x}=0}^{\mathrm{x}=L}$$

$$+ \int_{\mathrm{x}=0}^{\mathrm{x}=L} \big(q(\mathrm{x}) - EI\mathrm{w}''''(\mathrm{x})\big)\delta\mathrm{w}(\mathrm{x})\, dx = 0\,. \tag{5.91}$$

This equation must hold for all possible boundary conditions for Q and M and for all admissible variations $\delta\mathrm{w}$. Note that, for traction boundary conditions, $\delta\mathrm{w}$ and $\delta\mathrm{w}'$ can be chosen arbitrarily at the boundaries x = 0 and x = L, and δw can be chosen arbitrarily on x ∈ (0, L). Therefore,

$$EI\mathrm{w}''''(\mathrm{x}) = q(\mathrm{x})\,, \tag{5.92}$$

for every x ∈ (0, L), which is the same as in Eq. (5.73), and

$$Q(\mathrm{x}) = -EI\mathrm{w}'''(\mathrm{x})\,, \tag{5.93}$$

$$M(\mathrm{x}) = -EI\mathrm{w}''(\mathrm{x})\,, \tag{5.94}$$

at x = 0 and x = L. Now, let $0 \le L_1 \le L_2 \le L$ and consider a sub - body $\Delta\mathfrak{B}$ of the beam $\mathfrak{B}$ bounded by the planes $\mathrm{x} = L_1$, $\mathrm{x} = L_2$, $\mathrm{y} = \pm b$, $\mathrm{z} = \pm c$ (see Fig. 15). On requiring the virtual work principle (5.9) to hold true for every sub - body $\Delta\mathfrak{B} \subset \mathfrak{B}$, we find that Eq. (5.91) has to hold, with 0 and L replaced by L_1 and L_2, respectively. This in turn implies that Eqs. (5.93) and (5.94) must hold at $\mathrm{x} = L_1$ and $\mathrm{x} = L_2$. Since L_1 and L_2 have been chosen arbitrarily, it follows that Eqs. (5.93) and (5.94) must hold for every x ∈ [0, L], which is the same as in Eq. (5.74). Therefore, the virtual work principle for classical elasticity does not provide more relations than those established in the parts B), C) of this section. This is different in gradient elasticity, as we shall see below.

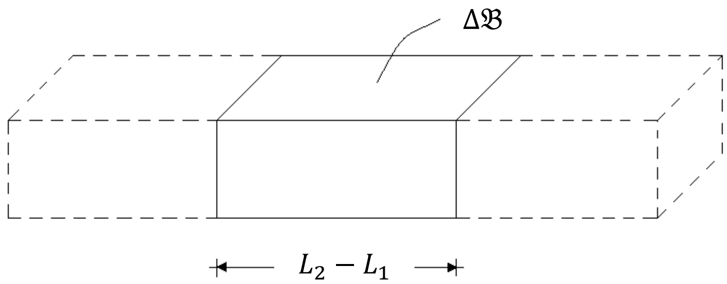


Figure 15: Sub - body $\Delta\mathfrak{B}$ of the beam $\mathfrak{B}$.

F) Distribution of the stresses τ and σ_x

We recall from engineering mechanics the formula for the shear stress

$$\tau = \tau(x,z) = \frac{Q(x)}{2I}(c^2 - z^2)\,. \tag{5.95}$$

This formula can be derived by considering a part $z \le \hat{z} \le c$ of the cross - section A (see Fig. 16). Indeed, it is readily proven, from Eqs. (5.30), (5.70), that

$$\sigma_x = \frac{M(x)}{I} z\,. \tag{5.96}$$

By taking the derivative with respect to x and utilizing Eq. (5.69),

$$\frac{\partial \sigma_x}{\partial x} = \frac{Q(x)}{I} z\,. \tag{5.97}$$

Combining this with Eq. (5.16),

$$\frac{Q(x)}{I} z + \frac{\partial \tau}{\partial z} = 0\,, \tag{5.98}$$

and after integration

$$\int_{\hat{z}=z}^{\hat{z}=c} \frac{\partial \tau}{\partial \hat{z}} d\hat{z} = -\frac{Q(x)}{I} \int_{\hat{z}=z}^{\hat{z}=c} \hat{z}\, d\hat{z}\,, \tag{5.99}$$

or

$$[\tau(x,\hat{z})]_{\hat{z}=z}^{\hat{z}=c} = \frac{Q(x)}{I}(z^2 - c^2)\,. \tag{5.100}$$

This, together with the boundary condition (5.54), furnishes Eq. (5.95).

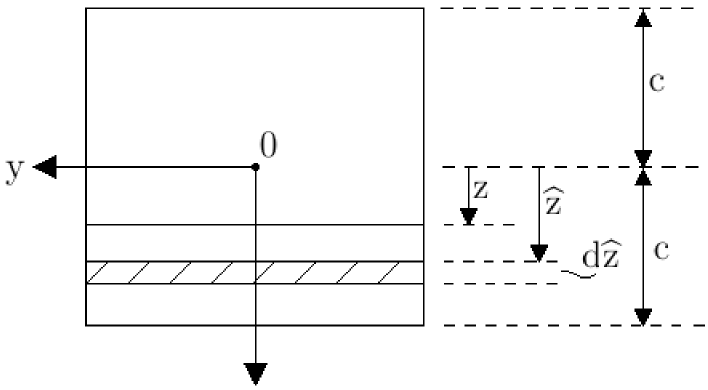


Figure 16: The part $z \leq \hat{z} \leq c$ of the cross - section A.

Having established the distribution of the shear stress (5.95), the distribution of σ_z can be calculated by substituting Eq. (5.95) into Eq. (5.17):

$$\frac{\partial \sigma_z}{\partial z} = -\frac{\partial \tau}{\partial x} = -\frac{Q'}{2I}(c^2 - z^2)\,. \tag{5.101}$$

After integration over z,

$$[\sigma_z]_c^z = -\frac{Q'}{2I}\left[c^2\hat{z} - \frac{\hat{z}^3}{3}\right]_{\hat{z}=c}^{\hat{z}=z}, \tag{5.102}$$

and by virtue of Eqs. (5.54), (5.64),

$$\sigma_z = \frac{q}{2I}\left(c^2 z - \frac{z^3}{3} - \frac{2}{3}c^3\right). \tag{5.103}$$

Note that this distribution satisfies the boundary condition (5.54).

5.2 A consistent formulation of the classical Euler - Bernoulli beam theory

As mentioned in the last section, there is an inconsistency between the assumed strain and stress states in the engineering mechanics approach, for the Euler - Bernoulli beam theory. In order to remove this inconsistency, and since it is useful for what follows, we shall rederive here the governing equations of the last section as a particular case of anisotropic elasticity. To elaborate, we assume stress and strain states of the form

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(x, z) \mathrel{\widehat{=}} \begin{pmatrix} \sigma_x & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & \sigma_z \end{pmatrix}, \tag{5.104}$$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(x, z) \mathrel{\widehat{=}} \begin{pmatrix} \epsilon_x & 0 & \epsilon_{xz} \\ 0 & 0 & 0 \\ \epsilon_{xz} & 0 & \epsilon_z \end{pmatrix}. \tag{5.105}$$

and that the equilibrium equations (5.16), (5.17),

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial z} = 0\,, \tag{5.106}$$

$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0\,, \tag{5.107}$$

hold.

From an axiomatic point of view, the Euler - Bernoulli beam theory might be justified by regarding the section forces as primitive variables, with Eq. (5.70) defining an elasticity law for M, the associated energy function being given in Eq. (5.82). In this setting, the kinematical variable is w = w(x). However, we will attempt here to justify the Euler - Bernoulli beam theory as limiting case of Eqs. (5.104) - (5.107). To this end, we recall from section 5.1.2 A), that according to the Euler - Bernoulli hypothesis, the cross - sections of the beam undergo rigid body motions. This is a kind of anisotropy and thus, we replace the isotropic elasticity law (5.5) with an anisotropic one

$$\Sigma_{ij} = \mathbb{K}_{ijmn}\epsilon_{mn}\,, \tag{5.108}$$

the associated free energy function being

$$\hat{\psi} = \frac{1}{2}\epsilon_{ij}\mathbb{K}_{ijmn}\epsilon_{mn}\,. \tag{5.109}$$

Here, $\mathbb{K}$ is an anisotropic elasticity tensor (fourth - order tensor exhibiting well - known symmetry properties, see the remarks after Eq. (3.33)). Assume the values $\mathbb{K}$ to be such, that for the problem at hand,

$$\Sigma_{11} = E\epsilon_{11}\,,\quad \Sigma_{13} = 2\hat{\mu}\epsilon_{13}\,,\quad \Sigma_{33} = \hat{E}\epsilon_{33}\,, \tag{5.110}$$

and all the other stress components being vanished, i.e., $\mathbb{K}_{1111} = E$, $\mathbb{K}_{1313} = \hat{\mu}$, $\mathbb{K}_{3333} = \hat{E}$. That means that $E, \hat{E}$ are Young's moduli and $\hat{\mu}$ is a shear modulus. The free energy in Eq. (5.109), for the problem at hand, reduces then to

$$\hat{\psi} = \frac{1}{2}\left[E(\epsilon_{11})^2 + \hat{E}(\epsilon_{33})^2 + 4\hat{\mu}(\epsilon_{13})^2\right]. \tag{5.111}$$

Now, we consider the Euler - Bernoulli beam theory as limiting case of Eqs. (5.104) - (5.110) for

$$\hat{E}\,,\ \ \hat{\mu} \longrightarrow \infty\,. \tag{5.112}$$

In order for the free energy $\hat{\psi}$ to remain bounded, which is a physical requirement,

$$\epsilon_{13}\,,\quad \epsilon_{33} \longrightarrow 0 \tag{5.113}$$

has to hold, implying that

$$\Sigma_{11} = E\epsilon_{11}\,, \tag{5.114}$$

$$\Sigma_{13}, \Sigma_{33}\text{: not determinable by the constitutive law}\,, \tag{5.115}$$

$$\hat{\psi} \longrightarrow \psi = \frac{1}{2}E\epsilon_x^2\,, \tag{5.116}$$

and hence (cf. Eq. (5.80))

$$\widehat{W}^{(i)} = \int_V \hat{\psi}\, dV \;\longrightarrow\; W^{(i)} = \frac{1}{2}\int_V E\, \epsilon_{\mathrm{x}}^2 dV \,. \tag{5.117}$$

Note, that although the strain ϵ_{z} disappears, a non - vanishing displacement w(x) in z - direction exists generally. In accordance with the Euler - Bernoulli beam theory we set (cf. (5.26), (5.24))

$$\mathbf{u} = \mathbf{u}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} -\mathrm{w}'(\mathrm{x})\,\mathrm{z} \\ 0 \\ \mathrm{w}(\mathrm{x}) \end{pmatrix}, \tag{5.118}$$

$$\epsilon_{11} = \epsilon_{\mathrm{x}} = -\mathrm{w}''(\mathrm{x})\,\mathrm{z}\,. \tag{5.119}$$

The remainder is the same as in sections 5.1.2 B) - 5.1.2 F).

5.3 Solution for specific boundary conditions

For the problem in Fig. 12 we assume

$$q(\mathrm{x}) = q_0 = \mathrm{const.} \tag{5.120}$$

and specify the boundary conditions (5.75) - (5.78) as follows

$$\mathrm{w}'(0) = 0\,, \qquad \mathrm{w}(0) = 0\,, \qquad M_L = 0\,, \qquad Q_L = 0\,, \tag{5.121}$$

or equivalently (cf. Eqs. (5.93), (5.94))

$$\mathrm{w}'(0) = 0\,, \qquad \mathrm{w}(0) = 0\,, \qquad \mathrm{w}''(L) = 0\,, \qquad \mathrm{w}'''(L) = 0\,. \tag{5.122}$$

The conditions (5.120), (5.121) reflect the cantilever beam sketched in Fig. 17.

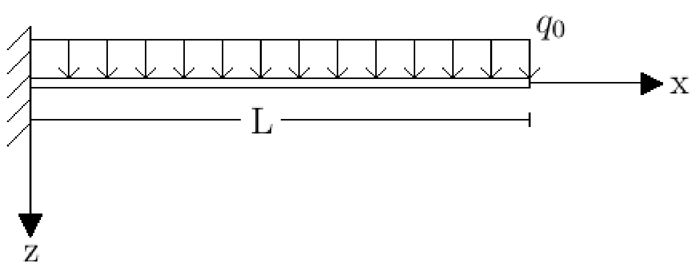


Figure 17: Cantilever beam subject to constant distributed load q_0.

For reasons of convenience, we introduce the dimensionless variables

$$\tilde{\mathrm{x}} := \frac{\mathrm{x}}{L}, \qquad \tilde{\mathrm{z}} := \frac{\mathrm{z}}{L}, \qquad \tilde{c} := \frac{c}{L}, \qquad \tilde{\mathrm{w}} := \frac{\mathrm{w}}{L}, \qquad \tilde{q}_0 := \frac{q_0}{EL},$$

$$\tilde{\sigma}_{\mathrm{x}} := \frac{\sigma_{\mathrm{x}}}{E}\,, \qquad \tilde{\tau} := \frac{\tau}{E}\,, \qquad \tilde{\sigma}_{\mathrm{z}} := \frac{\sigma_{\mathrm{x}}}{E}\,, \qquad \widetilde{M} := \frac{M}{EL^3}\,. \tag{5.123}$$

We have to solve the differential equation (5.73), which in dimensionless form becomes

$$\tilde{\mathrm{w}}'''' = \frac{3\tilde{q}_0}{4\tilde{b}\tilde{c}^3}. \tag{5.124}$$

The concomitant boundary conditions follow, from Eq. (5.122), to be

$$\tilde{\mathrm{w}}'(0) = 0,\ \ \tilde{\mathrm{w}}(0) = 0,\ \ \tilde{\mathrm{w}}''(1) = 0,\ \ \tilde{\mathrm{w}}'''(1) = 0\,. \tag{5.125}$$

The solution reads

$$\tilde{\mathrm{w}} = \tilde{c}_1 + \tilde{c}_2\tilde{\mathrm{x}} + \tilde{c}_3\tilde{\mathrm{x}}^2 + \tilde{c}_4\tilde{\mathrm{x}}^3 + \frac{1}{24}\frac{3\tilde{q}_0}{4\tilde{b}\tilde{c}^3}\tilde{\mathrm{x}}^4\,, \tag{5.126}$$

with constants of integration being

$$\tilde{c}_1 = \tilde{c}_2 = 0\,, \qquad \tilde{c}_3 = \frac{3\tilde{q}_0}{16\tilde{b}\tilde{c}^3}\,, \qquad \tilde{c}_4 = -\frac{3\tilde{q}_0}{24\tilde{b}\tilde{c}^3}\,. \tag{5.127}$$

Further, from Eqs. (5.74), (5.126), we obtain

$$\widetilde{M} = -\tilde{\mathrm{w}}'' = \tilde{q}_0\left(\tilde{\mathrm{x}} - \frac{1}{2}\tilde{\mathrm{x}}^2 - \frac{1}{2}\right)\,, \tag{5.128}$$

and we conclude, from Eqs. (5.96), (5.128), that

$$\tilde{\sigma}_{\mathrm{x}} = -\tilde{\mathrm{w}}''(\tilde{\mathrm{x}})\,\tilde{\mathrm{z}}\,, \tag{5.129}$$

$$\tilde{\sigma}_0(\tilde{\mathrm{z}}) \coloneqq \tilde{\sigma}_{\mathrm{x}}(\tilde{\mathrm{x}} = 0, \tilde{\mathrm{z}}) = -\tilde{\mathrm{w}}''(0)\tilde{\mathrm{z}}\,, \tag{5.130}$$

$$(\tilde{\sigma}_{\mathrm{x}})_{\max} \coloneqq \max\{|\tilde{\sigma}_{\mathrm{x}}(0, \tilde{\mathrm{z}})|\} = |\tilde{\sigma}_0(\tilde{\mathrm{z}} = \pm\tilde{c})|\,. \tag{5.131}$$

Now, to recast stress τ and σ_z in dimensionless form, we make use of Eqs. (5.95), (5.103) respectively, in conjunction with Eq. (5.93) and the definitions in Eq. (5.123) and we finally obtain

$$\tilde{\tau}(\tilde{\mathrm{x}}, \tilde{\mathrm{z}}) = -\frac{1}{2}\tilde{\mathrm{w}}'''(\tilde{\mathrm{c}}^2 - \tilde{\mathrm{z}}^2)\,, \tag{5.132}$$

$$\tilde{\sigma}_{\mathrm{z}}(\tilde{\mathrm{x}}, \tilde{\mathrm{z}}) = \frac{1}{2}\tilde{\mathrm{w}}''''\left(\tilde{\mathrm{c}}^2\tilde{\mathrm{z}} - \frac{\tilde{\mathrm{z}}^3}{3} - \frac{2}{3}\tilde{\mathrm{c}}^3\right)\,. \tag{5.133}$$

5.4 Euler - Bernoulli beam in the setting of gradient elasticity based on the anisotropic KG - Model

We shall try to develop a consistent Euler - Bernoulli beam theory by extending the anisotropic considerations of section 5.1.3 to gradient effects. The methodology relies essentially upon fundamental assumptions, definition of section forces and formulation of governing equations. The aim is, roughly speaking, to develop a theory compatible with the Euler - Bernoulli beam geometry by mainly satisfying equilibrium equations globally in terms of averages over cross - section areas, which we shall define appropriately.

A) Fundamental assumptions

As in section 5.1.3, we assume again stress and strain states of the forms

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} \Sigma_{11} & 0 & \Sigma_{13} \\ 0 & 0 & 0 \\ \Sigma_{13} & 0 & \Sigma_{33} \end{pmatrix}, \tag{5.134}$$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} \epsilon_{11} & 0 & \epsilon_{13} \\ 0 & 0 & 0 \\ \epsilon_{13} & 0 & \epsilon_{33} \end{pmatrix}. \tag{5.135}$$

Omitting body forces and acceleration terms, we obtain from Eqs. (3.28), (3.25), that

$$\partial_1 \Sigma_{11} + \partial_3 \Sigma_{13} = 0\,, \tag{5.136}$$

$$\partial_1 \Sigma_{13} + \partial_3 \Sigma_{33} = 0\,, \tag{5.137}$$

with

$$\Sigma_{jk} = \tau_{jk} - \partial_i \mu_{ijk}\,. \tag{5.138}$$

Generalizing the procedure for the classical case, we consider an anisotropic KG - Model, i.e., we assume for $\boldsymbol{\tau}$ in Eq. (3.30)

$$\tau_{jk} = \tau_{jk}(\mathrm{x}, \mathrm{y}) = \mathbb{K}_{jkmn} \epsilon_{mn}\,, \tag{5.139}$$

which is postulated to be equivalent to

$$\begin{pmatrix} \tau_{11} & 0 & \tau_{13} \\ 0 & 0 & 0 \\ \tau_{13} & 0 & \tau_{33} \end{pmatrix} = \begin{pmatrix} E\epsilon_{11} & 0 & 2\hat{\mu}\epsilon_{13} \\ 0 & 0 & 0 \\ 2\hat{\mu}\epsilon_{13} & 0 & \hat{E}\epsilon_{33} \end{pmatrix}. \tag{5.140}$$

Accordingly, for the stress $\boldsymbol{\mu}$ in Eq. (3.31), we set

$$\mu_{ijk} = l^2 \mathbb{K}_{jkmn} \partial_i \epsilon_{mn}\,. \tag{5.141}$$

Proceeding to calculate the terms in this equation, we note that

$$\partial_1 \epsilon_{mn} \mathrel{\widehat{=}} \begin{pmatrix} \partial_1 \epsilon_{11} & 0 & \partial_1 \epsilon_{13} \\ 0 & 0 & 0 \\ \partial_1 \epsilon_{13} & 0 & \partial_1 \epsilon_{33} \end{pmatrix}, \tag{5.142}$$

$$\partial_2 \epsilon_{mn} = 0\,, \tag{5.143}$$

$$\partial_3\epsilon_{mn} \mathrel{\widehat{=}} \begin{pmatrix} \partial_3\epsilon_{11} & 0 & \partial_3\epsilon_{13} \\ 0 & 0 & 0 \\ \partial_3\epsilon_{13} & 0 & \partial_3\epsilon_{33} \end{pmatrix}, \tag{5.144}$$

and hence

$$\mu_{1jk} = \mu_{1jk}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} l^2 \begin{pmatrix} E\,\partial_1\epsilon_{11} & 0 & 2\hat{\mu}\,\partial_1\epsilon_{13} \\ 0 & 0 & 0 \\ 2\hat{\mu}\,\partial_1\epsilon_{13} & 0 & \hat{E}\,\partial_1\epsilon_{33} \end{pmatrix}, \tag{5.145}$$

$$\mu_{2jk} = 0\,, \tag{5.146}$$

$$\mu_{3jk} = \mu_{3jk}(\mathrm{x}, \mathrm{z}) \mathrel{\widehat{=}} l^2 \begin{pmatrix} E\,\partial_3\epsilon_{11} & 0 & 2\hat{\mu}\,\partial_3\epsilon_{13} \\ 0 & 0 & 0 \\ 2\hat{\mu}\,\partial_3\epsilon_{13} & 0 & \hat{E}\,\partial_3\epsilon_{33} \end{pmatrix}. \tag{5.147}$$

The anisotropic version of the free energy in Eq. (3.29) reads

$$\hat{\psi} = \frac{1}{2}\epsilon_{ij}\mathbb{K}_{ijmn}\epsilon_{mn} + \frac{l^2}{2}k_{ijk}\mathbb{K}_{jkmn}k_{imn}\,, \tag{5.148}$$

which, for the assumptions made in this section, becomes (cf. Eq.(5.111))

$$\begin{aligned} \hat{\psi} = {} & \frac{1}{2}\left[E(\epsilon_{11})^2 + \hat{E}(\epsilon_{33})^2 + 4\hat{\mu}(\epsilon_{13})^2\right] \\ & + \frac{l^2}{2}\left[E(\partial_1\epsilon_{11})^2 + \hat{E}(\partial_1\epsilon_{33})^2 + 4\hat{\mu}(\partial_1\epsilon_{13})^2\right] \\ & + \frac{l^2}{2}\left[E(\partial_3\epsilon_{11})^2 + \hat{E}(\partial_3\epsilon_{33})^2 + 4\hat{\mu}(\partial_3\epsilon_{13})^2\right]. \end{aligned} \tag{5.149}$$

As in the classical case, we consider the limiting case of the equations above for

$$\hat{E}\,,\ \ \hat{\mu} \longrightarrow \infty\,. \tag{5.150}$$

It is a physical requirement, that the free energy should be bounded always. Thus, from Eq. (5.149),

$$\epsilon_{13}\,,\ \ \epsilon_{33}\,,\ \ \partial_i\epsilon_{13}\,,\ \ \partial_i\epsilon_{33}\ \longrightarrow\ 0 \tag{5.151}$$

must hold, implying

$$\epsilon_{ij} \longrightarrow \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.152}$$

$$\partial_1\epsilon_{jk} \longrightarrow \begin{pmatrix} \partial_1\epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.153}$$

$$\partial_2\epsilon_{jk} = 0\,, \tag{5.154}$$

$$\partial_3\epsilon_{jk} \longrightarrow \begin{pmatrix} \partial_3\epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.155}$$

In addition,

$$\tau_{ij} \longrightarrow \begin{pmatrix} E\epsilon_{11} & 0 & \tau_{13} \\ 0 & 0 & 0 \\ \tau_{13} & 0 & \tau_{33} \end{pmatrix} , \tag{5.156}$$

$$\mu_{1jk} \longrightarrow \begin{pmatrix} l^2 E\partial_1\epsilon_{11} & 0 & \mu_{113} \\ 0 & 0 & 0 \\ \mu_{113} & 0 & \mu_{133} \end{pmatrix} , \tag{5.157}$$

$$\mu_{2jk} = 0\,, \quad \mu_{ij2} = 0\,, \tag{5.158}$$

$$\mu_{3jk} \longrightarrow \begin{pmatrix} l^2 E\partial_3\epsilon_{11} & 0 & \mu_{313} \\ 0 & 0 & 0 \\ \mu_{313} & 0 & \mu_{333} \end{pmatrix} , \tag{5.159}$$

and

$$\Sigma_{jk} = \tau_{jk} - \partial_i\mu_{ijk} \longrightarrow \begin{pmatrix} E\epsilon_{11} - \partial_i\mu_{i11} & 0 & \tau_{13} - \partial_i\mu_{i13} \\ 0 & 0 & 0 \\ \tau_{13} - \partial_i\mu_{i13} & 0 & \tau_{33} - \partial_i\mu_{i33} \end{pmatrix} . \tag{5.160}$$

The stress components

$$\tau_{13}\,,\ \ \tau_{33}\,,\ \ \mu_{113}\,,\ \ \mu_{133}\,,\ \ \mu_{313}\,,\ \ \mu_{333}\,,\ \ \Sigma_{13}\,,\ \ \Sigma_{33}\,, \tag{5.161}$$

are not determinable from the constitutive laws.

Now, we suppose the Euler - Bernoulli beam geometry to hold, which imposes for the displacement **u** the form (cf. Eq. (5.118))

$$\mathbf{u} = \mathbf{u}(\mathrm{x},\mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} -\mathrm{w}'(\mathrm{x})\,\mathrm{z} \\ 0 \\ \mathrm{w}(\mathrm{x}) \end{pmatrix} , \tag{5.162}$$

and for the strain components the forms (cf. Eqs. (5.119), (5.152))

$$\epsilon_{ij} = \epsilon_{ij}(\mathrm{x},\mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} -\mathrm{w}''\mathrm{z} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{5.163}$$

It follows that

$$\partial_1\epsilon_{jk} \mathrel{\widehat{=}} \begin{pmatrix} -\mathrm{w}'''\mathrm{z} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{5.164}$$

$$\partial_3\epsilon_{jk} \mathrel{\widehat{=}} \begin{pmatrix} -\mathrm{w}'' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{5.165}$$

$$\tau_{ij} \mathrel{\widehat{=}} \begin{pmatrix} -E\mathrm{w}''\mathrm{z} & 0 & \tau_{13} \\ 0 & 0 & 0 \\ \tau_{13} & 0 & \tau_{33} \end{pmatrix}, \tag{5.166}$$

$$\mu_{1jk} \mathrel{\widehat{=}} \begin{pmatrix} -l^2 E\mathrm{w}'''\mathrm{z} & 0 & \mu_{113} \\ 0 & 0 & 0 \\ \mu_{113} & 0 & \mu_{133} \end{pmatrix}, \tag{5.167}$$

$$\mu_{3jk} \mathrel{\widehat{=}} \begin{pmatrix} -l^2 E\mathrm{w}'' & 0 & \mu_{313} \\ 0 & 0 & 0 \\ \mu_{313} & 0 & \mu_{333} \end{pmatrix}, \tag{5.168}$$

and

$$\Sigma_{jk} = \tau_{jk} - \partial_i \mu_{ijk} \mathrel{\widehat{=}} \begin{pmatrix} -E\mathrm{w}''\mathrm{z} + l^2 \mathrm{E}\mathrm{w}''''\mathrm{z} & 0 & \Sigma_{13} \\ 0 & 0 & 0 \\ \Sigma_{13} & 0 & \Sigma_{33} \end{pmatrix}$$

$$= \begin{pmatrix} -E\mathrm{w}''\mathrm{z} + l^2 E\mathrm{w}''''\mathrm{z} & 0 & \tau_{13} - \partial_1\mu_{113} - \partial_3\mu_{313} \\ 0 & 0 & 0 \\ \tau_{13} - \partial_1\mu_{113} - \partial_3\mu_{313} & 0 & \tau_{33} - \partial_1\mu_{133} - \partial_3\mu_{333} \end{pmatrix}. \tag{5.169}$$

Assume further that all derivatives of w(x), entering into the theory, are continuous on $[0, L]$ and hence bounded on this interval. We conclude from Eq. (5.169) that $\Sigma_{11}(\mathrm{x}, \mathrm{z})$ and also $\partial_1 \Sigma_{11}(\mathrm{x}, \mathrm{z})$ are continuous on the domain D* = $[0, L] \times [-c, c]$ and hence bounded on D*. From the equilibrium equations (5.136), (5.137), we conclude that $\Sigma_{13}(\mathrm{x,z})$ and $\Sigma_{33}(\mathrm{x,z})$ are continuously differentiable (and bounded) on D*. It follows from Eq. (5.169) that the stress τ_{13} , τ_{33} , μ_{113} , μ_{313} , μ_{133} , μ_{333} are continuously differentiable (and bounded) on D*. Therefore, keeping in mind Eqs. (5.140), (5.145) and (5.147), we can infer that the terms $\hat{\mu}\epsilon_{13}$, $\hat{E}\epsilon_{33}$, $\hat{\mu}\partial_1\epsilon_{13}$, $\hat{E}\partial_1\epsilon_{33}$, $\hat{\mu}\partial_3\epsilon_{13}$, $\hat{E}\partial_3\epsilon_{33}$ are bounded on D* for the limiting case (5.150). Finally, we find from Eq. (5.149), that

$$\hat{\psi} \to \psi = \frac{1}{2} E(\epsilon_{11})^2 + \frac{l^2}{2} E(\partial_1\epsilon_{11})^2 + \frac{l^2}{2}(\partial_3\epsilon_{11})^2 . \tag{5.170}$$

By substituting Eqs. (5.163) - (5.165) into Eq. (5.170), we obtain

$$\psi = \frac{1}{2} E(\mathrm{w}'')^2 \mathrm{z}^2 + \frac{l^2}{2} E(\mathrm{w}''')^2 \mathrm{z}^2 + \frac{l^2}{2} E(\mathrm{w}'')^2 . \tag{5.171}$$

B) Boundary conditions and cross - sections

Generally, on boundary planes and cross - sections

$$\partial_i \mathrm{n}_j = 0, \tag{5.172}$$

and from definitions (2.7), (2.8) we infer that

$$D_l \mathrm{n}_l = \partial_l \mathrm{n}_l - \mathrm{n}_l \mathrm{n}_k \, \partial_k \mathrm{n}_l = 0 , \tag{5.173}$$

$$D_j(\mathrm{n}_i \mu_{ijk}) = \mathrm{n}_i \, \partial_j \mu_{ijk} - \mathrm{n}_j \mathrm{n}_l \mathrm{n}_i \, \partial_l \mu_{ijk} . \tag{5.174}$$

On account of Eqs. (5.173), (5.174), the classical traction **P** in Eq. (3.40) becomes

$$P_k = \mathrm{n}_j \Sigma_{\mathrm{jk}} - \mathrm{n}_i\, \partial_j \mu_{ijk} + \mathrm{n}_j \mathrm{n}_l \mathrm{n}_i \partial_l \mu_{ijk}\,, \tag{5.175}$$

or, in view of Eqs. (5.160), (5.158),

$$\begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix} = \begin{pmatrix} \mathrm{n}_j \Sigma_{j1} - \mathrm{n}_i \partial_j \mu_{ij1} + \mathrm{n}_j \mathrm{n}_l \mathrm{n}_i \partial_l \mu_{ij1} \\ 0 \\ \mathrm{n}_j \Sigma_{j3} - \mathrm{n}_i \partial_j \mu_{ij3} + \mathrm{n}_j \mathrm{n}_l \mathrm{n}_i \partial_l \mu_{ij3} \end{pmatrix}. \tag{5.176}$$

For the non - classical traction **R** in Eq. (3.41), we get

$$\begin{pmatrix} R_1 \\ 0 \\ R_3 \end{pmatrix} = \begin{pmatrix} \mathrm{n}_i \mathrm{n}_j \mu_{ij1} \\ 0 \\ \mathrm{n}_i \mathrm{n}_j \mu_{ij3} \end{pmatrix}. \tag{5.177}$$

Also, for all boundary planes and cross sections, and for the problems in Fig. 12 (respectively 14), we can derive from Eq. (5.162) the normal derivative

$$D\mathrm{u}_k = \mathrm{n}_l \partial_l \mathrm{u}_k \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{n}_l \partial_l \mathrm{u}_1 \\ 0 \\ \mathrm{n}_l \partial_l \mathrm{u}_3 \end{pmatrix} = \begin{pmatrix} -\mathrm{n}_1 \mathrm{w}''\mathrm{z} - \mathrm{n}_3 \mathrm{w}' \\ 0 \\ \mathrm{n}_1 \mathrm{w}' \end{pmatrix}, \tag{5.178}$$

and the variations

$$\delta \mathbf{u} \mathrel{\widehat{=}} \begin{pmatrix} -\delta \mathrm{w}'\mathrm{z} \\ 0 \\ \delta \mathrm{w} \end{pmatrix}, \tag{5.179}$$

$$\delta(D\mathrm{u}_\mathrm{k}) = D\delta \mathrm{u}_\mathrm{k} \mathrel{\widehat{=}} \begin{pmatrix} -\mathrm{n}_1 \delta \mathrm{w}''\mathrm{z} - \mathrm{n}_3 \delta \mathrm{w}' \\ 0 \\ \mathrm{n}_1 \delta \mathrm{w}' \end{pmatrix}, \tag{5.180}$$

C) Section forces

In the same manner as in part B) of section 5.1.2, we may introduce appropriate section forces by considering the virtual work on a boundary plane A of a part of the beam $\Delta\mathfrak{B}$. On such planes

$$\mathbf{n} = \pm \mathbf{e}_\mathrm{x} \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{n}_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}, \tag{5.181}$$

and from Eqs. (5.176), (5.177), (5.179), (5.180),

$$\begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix}_A = \mathrm{n}_1 \begin{pmatrix} \Sigma_{11} - \partial_j \mu_{1j1} + \partial_1 \mu_{111} \\ 0 \\ \Sigma_{13} - \partial_j \mu_{1j3} + \partial_1 \mu_{113} \end{pmatrix}_A = \mathrm{n}_1 \begin{pmatrix} \Sigma_{11} - \partial_3 \mu_{131} \\ 0 \\ \Sigma_{13} - \partial_3 \mu_{133} \end{pmatrix}_A, \tag{5.182}$$

$$\begin{pmatrix} R_1 \\ 0 \\ R_3 \end{pmatrix}_A = \begin{pmatrix} \mu_{111} \\ 0 \\ \mu_{113} \end{pmatrix}_A, \tag{5.183}$$

$$[\delta\mathbf{u}]_A \triangleq \begin{pmatrix} -\delta \mathrm{w}'\mathrm{z} \\ 0 \\ \delta \mathrm{w} \end{pmatrix}_A \quad , \tag{5.184}$$

$$[\delta(D\mathbf{u})]_A \triangleq \mathrm{n}_1 \begin{pmatrix} -\delta \mathrm{w}''\mathrm{z} \\ 0 \\ \delta \mathrm{w}' \end{pmatrix}_A \quad . \tag{5.185}$$

The virtual work expended by the external tractions **P** and **R** is suggested by the first integral in the left side of Eq. (3.42). Therefore, using Eqs. (5.182) - (5.185), the virtual work expended by **P** and **R** on A, for the problems in Fig. 12, is given by

$$\int_A \mathbf{P}\cdot\delta\mathbf{u}\, dS + \int_A \mathbf{R}\cdot\delta(D\mathbf{u})dS = \int_A \begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix}\cdot\begin{pmatrix} -\delta \mathrm{w}'\mathrm{z} \\ 0 \\ \delta \mathrm{w} \end{pmatrix} dS \ + \int_A \begin{pmatrix} R_1 \\ 0 \\ R_3 \end{pmatrix}\cdot\begin{pmatrix} -\delta \mathrm{w}''\mathrm{z} \\ 0 \\ \delta \mathrm{w}' \end{pmatrix} \mathrm{n}_1 dS$$

$$= \left(\int_A P_1 \mathrm{z} dS\right)(-\delta\mathrm{w}') + \left(\int_A P_3 dS\right)\delta\mathrm{w} + \left(\int_A R_1 \mathrm{z} dS\right)\mathrm{n}_1(-\delta\mathrm{w}'') + \left(\int_A R_3 dS\right)\mathrm{n}_1\delta\mathrm{w}'$$

$$= \left[\int_A (\Sigma_{11} - \partial_3\mu_{131})\mathrm{z}\, dS - \int_A \mu_{113}\, dS\right]\mathrm{n}_1(-\delta\mathrm{w}') + \left[\int_A (\Sigma_{13} - \partial_3\mu_{133})\, dS\right]\mathrm{n}_1\delta\mathrm{w}$$

$$= \left[\int_A \mu_{111}\mathrm{z}\, dS\right]\mathrm{n}_1(-\delta\mathrm{w}'') \, . \tag{5.186}$$

As $\delta\mathrm{w}, \delta\mathrm{w}', \delta\mathrm{w}''$ can be chosen arbitrarily on the boundary planes A, we can introduce section displacement vector **f**, section angular vector $\boldsymbol{\phi}$ and section curvature vector **c** by

$$\mathbf{f} = \mathbf{f}(\mathrm{x}) = \mathrm{w}\mathbf{e}_\mathrm{z} \, , \tag{5.187}$$

$$\boldsymbol{\phi} = \boldsymbol{\phi}(\mathrm{x}) = (-\mathrm{w}')\mathbf{e}_\mathrm{y} \, , \tag{5.188}$$

$$\mathbf{c} = \mathbf{c}(\mathrm{x}) = (-\mathrm{w}'')\mathbf{e}_\mathrm{z} \, . \tag{5.189}$$

Recall that for small deformations, we deal with, the curvature of the flexure curve w(x) is approximated by w'' and that the direction vector at x, along the radius of the curvature circle at that point, is approximated by $\mathbf{e}_\mathrm{z}$. This is the reason why we have assumed **c** in Eq. (5.189) to be coaxial to $\mathbf{e}_\mathrm{z}$. Then, Eq. (5.186) suggests defining section shearing force vector **Q**, section bending moment vector **M** and section non - classical moment vector **m**, by (cf. Eqs. (5.36), (5.37) for the classical case)

$$\mathbf{Q} = \mathbf{Q}(\mathbf{n}, \mathrm{x}) = Q(\mathrm{x})\mathrm{n}_1\mathbf{e}_\mathrm{z} \, , \tag{5.190}$$

$$\mathbf{M} = \mathbf{M}(\mathbf{n}, \mathrm{x}) = M(\mathrm{x})\mathrm{n}_1\mathbf{e}_\mathrm{y} \, , \tag{5.191}$$

$$\mathbf{m} = \mathbf{m}(\mathbf{n}, \mathrm{x}) = m(\mathrm{x})\mathrm{n}_1\mathbf{e}_\mathrm{z} \, , \tag{5.192}$$

with the scalar section forces Q, M, m being

$$Q = Q(\mathrm{x}) = \int_A (\Sigma_{13} - \partial_3\mu_{133})dS, \tag{5.193}$$

$$M = M(\mathrm{x}) = \int_A (\Sigma_{11} - \partial_3\mu_{131})\mathrm{z}dS - \int_A \mu_{113} dS$$

$$= \int_A \Sigma_{11} \mathrm{z} dS - \int_A \partial_3(\mu_{131}\mathrm{z}) dS \,, \tag{5.194}$$

$$m = m(\mathrm{x}) = \int_A \mu_{111} \mathrm{z} dS . \tag{5.195}$$

It is worth noticing, that Eq. (5.195) indicates that m is a first order moment of μ_{111} . We conclude that the virtual work expended by **P** and **R** on a section surface A with outward normal **n** may be recast in the form

$$\int_A \mathbf{P} \cdot \delta \mathbf{u} dS + \int_A \mathbf{R} \cdot \delta(\mathrm{D}\mathbf{u}) dS = [Q(\mathrm{x})\mathrm{n}_1]\delta \mathrm{w}(\mathrm{x}) + [M(\mathrm{x})\mathrm{n}_1]\big(-\delta \mathrm{w}'(\mathrm{x})\big) + [m(\mathrm{x})\mathrm{n}_1]\big(-\delta \mathrm{w}''(\mathrm{x})\big) . \tag{5.196}$$

To accomplish the section forces we define the section normal force vectors (see Appendix A)

$$\mathbf{N} = \mathbf{N}(\mathbf{n}, \mathrm{x}) = \int_A P_1 dS \, \mathbf{e}_\mathrm{x} = \int_A (\Sigma_{11} - \partial_3 \mu_{131}) dS \;\, \mathrm{n}_1 \mathbf{e}_\mathrm{x} = N \mathrm{n}_1 \mathbf{e}_\mathrm{x} \equiv N \mathbf{n} \,, \tag{5.197}$$

$$\mathbf{N}_R = \mathbf{N}_R(\mathbf{n}, \mathrm{x}) = \int_A R_1 dS \;\, \mathrm{n}_1 \mathbf{e}_\mathrm{x} = \int_A \mu_{111} dS \;\, \mathrm{n}_1 \mathbf{e}_\mathrm{x} = N_R \mathrm{n}_1 \mathbf{e}_\mathrm{x} \equiv N_R \mathbf{n} \,, \tag{5.198}$$

where the scalar normal section forces N, N_R are given by

$$N = N(\mathrm{x}) = \int_A (\Sigma_{11} - \partial_3 \mu_{131}) dS \,, \tag{5.199}$$

$$N_R = N_R(\mathrm{x}) = \int_A \mu_{111} dS . \tag{5.200}$$

Note that, by Eqs. (5.158), (5.167) - (5.169), (5.71),

$$\int_A \Sigma_{11} \mathrm{z} dS = -E \int_A (\mathrm{w}''\mathrm{z}^2 - l^2 \mathrm{w}''''\mathrm{z}^2) dS = -EI\mathrm{w}'' + l^2 EI \mathrm{w}'''' \,, \tag{5.201}$$

$$\int_A \mu_{111} dS = -l^2 E \mathrm{w}''' \int_A \mathrm{z} dS , \tag{5.202}$$

$$\int_A \mu_{111} \mathrm{z} dS = -l^2 E \mathrm{w}''' \int_A \mathrm{z}^2 dS = -l^2 EI \mathrm{w}''' \,, \tag{5.203}$$

$$\begin{aligned}
\int_A \big(\Sigma_{11} - \partial_3 \mu_{131}\big) \, dS &= \int_A [E(-\mathrm{w}'' + l^2 \mathrm{w}'''')\mathrm{z} - \partial_3 \mu_{131}] dS \\
&= E(-\mathrm{w}'' + l^2 \mathrm{w}'''') \int_A \mathrm{z} dS - \int_{\mathrm{y}=-\mathrm{b}}^{\mathrm{y}=\mathrm{b}} \int_{\mathrm{z}=-\mathrm{c}}^{\mathrm{z}=\mathrm{c}} \partial_\mathrm{z} \mu_{131} dy \, dz \\
&= E(-\mathrm{w}'' + l^2 \mathrm{w}'''') \int_A \mathrm{z} dS - 2b[\mu_{131}]_{\mathrm{z}=-\mathrm{c}}^{\mathrm{z}=\mathrm{c}} .
\end{aligned} \tag{5.204}$$

Thus, Eqs. (5.193) - (5.195), (5.199) and (5.200) can be rewritten in the forms

$$Q = Q(\mathrm{x}) = \int_A \Sigma_{13} dS - \int_A \partial_3 \mu_{133} dS, \tag{5.205}$$

$$M = M(\mathrm{x}) = -EI\mathrm{w}'' + l^2 EI\mathrm{w}'''' - \int_A \partial_{\mathrm{z}}(\mu_{131}\mathrm{z}) dS, \tag{5.206}$$

$$m = m(\mathrm{x}) = -l^2 EI\mathrm{w}''' , \tag{5.207}$$

$$N = N(\mathrm{x}) = -E(\mathrm{w}'' - l^2\mathrm{w}'''') \int_A \mathrm{z} dS - 2b[\mu_{131}]_{z=-c}^{z=c} , \tag{5.208}$$

$$N_R = N_R(\mathrm{x}) = -l^2 E\mathrm{w}''' \int_A \mathrm{z} dS. \tag{5.209}$$

As in the classical case, we shall formulate the remaining part of the theory in terms of the scalar section forces.

D) Boundary conditions

Similar to the classical case, we shall formulate the boundary conditions in mixed form, i.e., we shall postulate the boundary conditions in both local and global forms. Globally formulated boundary conditions will be expressed in terms of section forces Q, M, m, N, N_R and/or section displacements w, w′, w″. This concerns only the boundary planes A① and A②. In the remaining boundary planes, the boundary conditions will be formulated locally. Only traction boundary conditions will be imposed on these planes, expressed in terms of tractions P and R in Eqs. (5.176), (5.177).

Plane A①, A②

$$\mathrm{x} = L\,, 0\,, \qquad \mathbf{n} = \pm\mathbf{e}_{\mathrm{x}} \mathrel{\widehat{=}} \begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.210}$$

By analogy to the specification of the boundary conditions on A①, A②. in section 5.1.2, we postulate for the beam in Fig. 12, that

$$N_L = N(L) = 0\,, \tag{5.211}$$

$$(N_R)_L = N_R(L) = 0\,, \tag{5.212}$$

$$(N_R)_0 = N_R(0) = 0\,, \tag{5.213}$$

and

$$\text{either } Q_L = Q(L) \text{ or } \mathrm{w}(L)\,, \tag{5.214}$$

$$\text{either } Q_0 = Q(0) \text{ or } \mathrm{w}(0)\,, \tag{5.215}$$

$$\text{either } M_L = M(L) \text{ or } \mathrm{w}'(L)\,, \tag{5.216}$$

$$\text{either } M_0 = M(0) \text{ or } \mathrm{w}'(0)\,, \tag{5.217}$$

$$\text{either } m_L = m(L) \text{ or } \mathrm{w}''(L)\,, \tag{5.218}$$

$$\text{either } m_0 = m(0) \text{ or } \mathrm{w}''(0) \tag{5.219}$$

have to be given.

Planes A⑤, A⑥

$$\mathrm{z} = \pm c\,, \qquad \mathbf{n} = \pm \mathbf{e}_{\mathrm{z}} \mathrel{\widehat{=}} \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}. \tag{5.220}$$

With respect to Eqs. (5.176), (5.177), we postulate in local form the classical boundary conditions

$$\begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix}_{\mathrm{z=c}} = \begin{pmatrix} \Sigma_{31} - \partial_1 \mu_{311} \\ 0 \\ \Sigma_{33} - \partial_1 \mu_{313} \end{pmatrix}_{\mathrm{z=c}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.221}$$

$$\begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix}_{\mathrm{z=-c}} = -\begin{pmatrix} \Sigma_{31} - \partial_1 \mu_{311} \\ 0 \\ \Sigma_{33} - \partial_1 \mu_{313} \end{pmatrix}_{\mathrm{z=-c}} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{q(x)}{2b} \end{pmatrix}, \tag{5.222}$$

and the non - classical boundary conditions

$$\begin{pmatrix} R_1 \\ 0 \\ R_3 \end{pmatrix}_{\mathrm{z=\pm c}} = \begin{pmatrix} \mu_{331} \\ 0 \\ \mu_{333} \end{pmatrix}_{\mathrm{z=\pm c}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.223}$$

Because of the last equation, the derivative $\partial_1 \mu_{313} \equiv \partial_1 \mu_{331}$ vanishes at $\mathrm{z} = \pm c$,

$$[\partial_1 \mu_{313}]_{\mathrm{z}=\pm c} = 0\,, \tag{5.224}$$

and taking into account Eq. (5.168), we conclude from Eqs. (5.221), (5.222) that

$$\begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix}_{\mathrm{z=c}} = \begin{pmatrix} \Sigma_{31} + l^2 E \mathrm{w}''' \\ 0 \\ \Sigma_{33} \end{pmatrix}_{\mathrm{z=c}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.225}$$

$$\begin{pmatrix} P_1 \\ 0 \\ P_3 \end{pmatrix}_{\mathrm{z=-c}} = -\begin{pmatrix} \Sigma_{31} + l^2 E \mathrm{w}''' \\ 0 \\ \Sigma_{33} \end{pmatrix}_{\mathrm{z=-c}} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{q(x)}{2b} \end{pmatrix}, \tag{5.226}$$

It follows from the last two equations, that

$$[\Sigma_{31}]_{\mathrm{z=\pm c}} = -l^2 E \mathrm{w}'''\,, \tag{5.227}$$

$$[\Sigma_{33}]_{\mathrm{z=-c}}^{\mathrm{z=c}} = \frac{q(\mathrm{x})}{2b}\,. \tag{5.228}$$

Planes A③, A④

$$\mathrm{y} = \pm b\,, \qquad \mathbf{n} = \pm \mathbf{e}_{\mathrm{y}} \mathrel{\widehat{=}} \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}. \tag{5.229}$$

These lateral planes are postulated to be traction - free with respect to both tractions **P** and **R**, in agreement with the assumed component forms in Eqs. (5.166) - (5.169), (5.158). This corresponds to boundary conditions in local form.

Before going to establish the governing equations, it is convenient to determine first the centroidal axis.

E) Governing equations

According to [10], the KG - Model may be viewed as a non - classical constitutive law for the classical Cauchy stress $\boldsymbol{\Sigma}$, with $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ denoting internal (variables) stresses (cf. Eq. (5.138)). The underlying continuum obeys the classical equilibrium equations (5.136), (5.137), and a virtual work principle of the form (5.9), which we shall formulate in the following. The goal is now to derive from the field equations the governing equations for the beam in Fig. 12. The procedure we shall pursue is merely the same we have employed in the last section for the classical case.

We start with the boundary condition (5.211), and assuming no axial external load to apply (see [19]), we conclude that

$$N'(\mathrm{x}) = 0\,, \tag{5.230}$$

$$N(\mathrm{x}) = 0\,, \tag{5.231}$$

everywhere on $[0, L]$. Using similar arguments, we infer from the boundary conditions (5.212) and (5.213), that

$$N_R'(\mathrm{x}) = 0\,, \tag{5.232}$$

$$N_R(\mathrm{x}) = 0\,, \tag{5.233}$$

everywhere on $[0, L]$.

Next, integrate the local equilibrium equation (5.136) over A,

$$\frac{d}{d\mathrm{x}}\int_A \Sigma_{11}\,dS + \int_{\mathrm{y}=-b}^{\mathrm{y}=b}\int_{\mathrm{z}=-c}^{\mathrm{z}=c} \frac{\partial \Sigma_{13}}{\partial \mathrm{z}}\,d\mathrm{z}\,d\mathrm{y} = 0\,, \tag{5.234}$$

or

$$\frac{d}{d\mathrm{x}}\int_A \Sigma_{11}\,dS + 2b[\Sigma_{13}]_{\mathrm{z}=-c}^{\mathrm{z}=c} = 0\,. \tag{5.235}$$

Because of condition (5.227),

$$[\Sigma_{13}]_{\mathrm{z}=-c}^{\mathrm{z}=c} = 0\,, \tag{5.236}$$

so that

$$\int_A \Sigma_{11}\,dS = k = \text{const.}\,, \tag{5.237}$$

and by virtue of Eq. (5.169),

$$-E(\mathrm{w}'' - l^2\mathrm{w}'''') \int_A \mathrm{z} dS = k\,. \tag{5.238}$$

It is immediately seen from Eqs. (5.208), (5.231) that

$$N = N(\mathrm{x}) = k - 2b[\mu_{131}]_{\mathrm{z}=-c}^{\mathrm{z}=c} = 0 \iff k = 2b[\mu_{131}]_{\mathrm{z}=-c}^{\mathrm{z}=c}\,. \tag{5.239}$$

Turning back again to the equilibrium equations, we integrate Eq. (5.137) over A, at arbitrary position x,

$$\int_A \frac{\partial \Sigma_{13}}{\partial \mathrm{x}} dS + \int_A \frac{\partial \Sigma_{33}}{\partial \mathrm{z}} dS = 0\,, \tag{5.240}$$

which yields

$$\frac{d}{d\mathrm{x}} \int_A \Sigma_{13} dS + 2b[\Sigma_{33}]_{\mathrm{z}=-\mathrm{c}}^{\mathrm{z}=\mathrm{c}} = 0\,. \tag{5.241}$$

The first term in this equation can be recast by using Eq. (5.205), while the second term can be rewritten with the help of Eq. (5.228),

$$Q' + \frac{d}{d\mathrm{x}} \int_A \partial_3 \mu_{133} dS + q = 0\,. \tag{5.242}$$

Equation (5.237), (respectively Eq. (5.239)) and Eq. (5.242) are the global forms of the equilibrium equations for forces. For deriving the corresponding equations for moments, we multiply (5.136) by z and integrate the result over A,

$$\int_A \frac{\partial \Sigma_{11}}{\partial \mathrm{x}} \mathrm{z} dS + \int_A \frac{\partial \Sigma_{13}}{\partial \mathrm{z}} \mathrm{z} dS = 0\,, \tag{5.243}$$

from which

$$\frac{d}{d\mathrm{x}} \int_A \Sigma_{11} \mathrm{z} dS + \int_A \frac{\partial(\Sigma_{13}\mathrm{z})}{\partial \mathrm{z}} dS - \int_A \Sigma_{13} dS = 0\,. \tag{5.244}$$

The second integral can be evaluated with the aid of Eq. (5.227),

$$\int_A \frac{\partial(\Sigma_{13}\mathrm{z})}{\partial \mathrm{z}} dS = \int_{\mathrm{y}=-\mathrm{b}}^{\mathrm{y}=\mathrm{b}} [\Sigma_{13}\mathrm{z}]_{\mathrm{z}=-\mathrm{c}}^{\mathrm{z}=\mathrm{c}}\, d\mathrm{y} = 2b([\Sigma_{13}\mathrm{z}]_{\mathrm{z}=\mathrm{c}} - [\Sigma_{13}\mathrm{z}]_{\mathrm{z}=-\mathrm{c}})$$

$$= 2bc([\Sigma_{13}]_{\mathrm{z}=\mathrm{c}} + [\Sigma_{13}]_{\mathrm{z}=-\mathrm{c}}) = -l^2 EA\mathrm{w}'''\,. \tag{5.245}$$

The term EA is known as the **axial rigidity** of the beam. The first and the third integral in Eq. (5.244) can be replaced with the help of Eqs. (5.193), (5.194). Altogether,

$$M' + \frac{d}{d\mathrm{x}} \int_A \partial_\mathrm{z}(\mu_{131}\mathrm{z}) dS - l^2 EA\mathrm{w}''' - \int_A \partial_\mathrm{z} \mu_{133} dS - Q = 0\,. \tag{5.246}$$

In Eqs. (5.239), (5.242) and (5.246), the terms containing components of the internal stress $\boldsymbol{\mu}$ have yet to be specified from the virtual work principle, which we are now going to formulate.

Quite similar to classical elasticity, the virtual work principle for gradient elasticity requires for the virtual work expended by external forces, $\delta W^{(e)}$, to be equal to the one expended by internal forces, $\delta W^{(i)}$, the latter being the same as the virtual change of the energy stored in the material. That means,

$$\delta W^{(e)} = \delta W^{(i)}\,, \tag{5.247}$$

with

$$\begin{aligned} W^{(i)} &= \int_V \psi dV = \frac{1}{2} E \int_{\mathrm{A}} \int_{\mathrm{x=0}}^{\mathrm{x=L}} [(\mathrm{w}'')^2 \mathrm{z}^2 + l^2 (\mathrm{w}''')^2 \mathrm{z}^2 + l^2 (\mathrm{w}'')^2] dA\, d\mathrm{x} \\ &= \frac{1}{2}(EI + l^2 EA) \int_0^L (\mathrm{w}'')^2 d\mathrm{x} + \frac{1}{2} l^2 EI \int_0^L (\mathrm{w}''')^2 d\mathrm{x}\,, \end{aligned} \tag{5.248}$$

where use has been made of Eq. (5.171) and the definition of moment of inertia in Eq. (5.71). Equation (5.248) states that the work of internal forces, besides the flexural rigidity EI, accounts also for the axial rigidity of the beam EA. After some algebraic manipulations, using repeatedly partial integration, we can infer from Eq. (5.248), that

$$\begin{aligned} \delta W^{(i)} &= \left[-\left((EI + l^2 EA)\mathrm{w}''' - l^2 EI \mathrm{w}'''''\right)\delta \mathrm{w}\right]_{\mathrm{x=0}}^{\mathrm{x=L}} \\ &+ \left[-\left((EI + l^2 EA)\mathrm{w}'' - l^2 EI \mathrm{w}''''\right)(-\delta \mathrm{w}')\right]_{\mathrm{x=0}}^{\mathrm{x=L}} \\ &+ [-l^2 EI \mathrm{w}'''(-\delta \mathrm{w}'')]_{\mathrm{x=0}}^{\mathrm{x=L}} + \int_0^{\mathrm{L}} [(EI + l^2 EA)\mathrm{w}'''' - l^2 EI \mathrm{w}''''''] \,\delta \mathrm{w}\, dx\,. \end{aligned} \tag{5.249}$$

In what concerns the virtual work of the external forces **P** and **R**, we conclude from Eq. (3.42) that

$$\delta W^{(e)} = \int_{\partial \mathrm{V}} \big(\mathbf{P} \cdot \delta \mathbf{u} + \mathbf{R} \cdot \delta(D\mathbf{u})\big) dA\,. \tag{5.250}$$

Moreover, from the boundary conditions imposed in part D) of this section, we see that the boundary planes $\mathrm{A}^{③}$, $\mathrm{A}^{④}$ and $\mathrm{A}^{⑤}$ are free of external tractions, so that only the tractions on the planes $\mathrm{A}^{①}$, $\mathrm{A}^{②}$ and $\mathrm{A}^{⑥}$ will contribute to $\delta W^{(e)}$. Therefore,

$$\delta W^{(e)} = \int_{\mathrm{A}^{①} \cup \mathrm{A}^{②} \cup \mathrm{A}^{⑥}} \big(\mathbf{P} \cdot \delta \mathbf{u} + \mathbf{R} \cdot \delta(D\mathbf{u})\big) dA\,. \tag{5.251}$$

As the boundary conditions on $\mathrm{A}^{①}$ and $\mathrm{A}^{②}$ are formulated globally, whereas on $\mathrm{A}^{⑥}$ they are imposed locally, we may establish for $\mathrm{A}^{①}$ and $\mathrm{A}^{②}$ the results, (cf. Eq. (5.196)),

$$\begin{aligned} \mathrm{x} = \mathrm{L}, \mathrm{n}_1 = 1: \int_{\mathrm{A}^{①}} \big(\mathbf{P} \cdot \delta \mathbf{u} + \mathbf{R} \cdot \delta(D\mathbf{u})\big) dS &= Q(L)\delta \mathrm{w}(L) + M(L)\big(-\delta \mathrm{w}'(L)\big) \\ &+ m(L)\big(-\delta \mathrm{w}''(L)\big)\,, \end{aligned} \tag{5.252}$$

$$\begin{aligned} \mathrm{x} = 0, \mathrm{n}_1 = -1: \int_{\mathrm{A}^{②}} \big(\mathbf{P} \cdot \delta \mathbf{u} + \mathbf{R} \cdot \delta(D\mathbf{u})\big) dS &= -Q(0)\delta \mathrm{w}(0) - M(0)\big(-\delta \mathrm{w}'(0)\big) \\ &- m(0)\big(-\delta \mathrm{w}''(0)\big)\,, \end{aligned} \tag{5.253}$$

and for $A^{⑥}$, the result, (cf. Eqs. (5.184), (5.223) and (5.226)),

$$\int_{A^{⑥}} \big(\mathbf{P}\cdot\delta\mathbf{u}+\mathbf{R}\cdot\delta(D\mathbf{u})\big)dS=\int_{A^{⑥}}(P_3\mathbf{e}_z)\cdot(\delta \mathrm{w}\mathbf{e}_z)dS$$

$$=\int_{x=0}^{x=L}\int_{y=-b}^{y=b}\frac{q(x)}{2b}\delta\mathrm{w}(x)dy\,dx=\int_0^L q(x)\,\delta\mathrm{w}(x)\,dx\,. \tag{5.254}$$

By summing the contribution (5.252) - (5.254),

$$\delta W^{(e)}=\big[Q(x)\delta\mathrm{w}(x)+M(x)\big(-\delta\mathrm{w}'(x)\big)+m(x)\big(-\delta\mathrm{w}''(x)\big)\big]_{x=0}^{x=L}+\int_0^L q(x)\delta\mathrm{w}(x)\,dx\,. \tag{5.255}$$

This way, we obtain from Eqs. (5.247), (5.249) and (5.255)

$$\begin{aligned}&[(Q(x)+(EI+l^2EA)\mathrm{w}'''-l^2EI\mathrm{w}''''')\delta\mathrm{w}(x)]_{x=0}^{x=L}\\&+\big[(M(x)+(EI+l^2EA)\mathrm{w}''-l^2EI\mathrm{w}'''')\big(-\delta\mathrm{w}'(x)\big)\big]_{x=0}^{x=L}\\&+\big[(m(x)+l^2EI\mathrm{w}''')\big(-\delta\mathrm{w}''(x)\big)\big]_{x=0}^{x=L}\\&+\int_0^L\big[q(x)-\big((EI+l^2EA)\mathrm{w}''''(x)-l^2EI\mathrm{w}''''''(x)\big)\big]\delta\mathrm{w}(x)\,dx=0\,.\end{aligned} \tag{5.256}$$

Using quite similar arguments as for the classical case in part E) of section 5.1.2, we may infer that

$$Q(x)=-[(EI+l^2EA)\mathrm{w}'''(x)-l^2EI\mathrm{w}'''''(x)]\,, \tag{5.257}$$

$$M(x)=-[(EI+l^2EA)\mathrm{w}''(x)-l^2EI\mathrm{w}''''(x)]\,, \tag{5.258}$$

$$m(x)=-l^2EI\mathrm{w}'''(x) \tag{5.259}$$

for $x\in[0,L]$, and that

$$(EI+l^2EA)\mathrm{w}''''(x)-l^2EI\mathrm{w}''''''(x)=q(x) \tag{5.260}$$

for $x\in(0,L)$.

Furthermore, it is readily shown from Eqs. (5.257), (5.258), (5.260), that

$$M'-Q=0\,, \tag{5.261}$$

$$Q'+q=0. \tag{5.262}$$

Equations (5.230), (5.232), (5.260) - (5.262) are the main governing field equations for the Euler - Bernoulli beam in the context of gradient elasticity based on the KG - Model. It is worth noticing that now, in contrast to the classical case, the virtual work principle provides further relations, besides the ones obtained from the equilibrium equations, namely the relations (5.257) - (5.260). We shall employ these relations in the subsequent parts F), G) in order to derive the final results.

F) Centroidal axis

Evidently, for given boundary conditions (5.211) - (5.219), a unique solution w(x) will be derived from Eqs. (5.257) - (5.260). Consequently, $\mathrm{w}''(\mathrm{x}) - l^2\mathrm{w}''''(\mathrm{x})$ in Eq. (5.238) will be a known function of x, implying that Eq. (5.238) will be satisfied for every x ∈ [0, L] if and only if

$$\int_A \mathrm{z}\, dS = k = 0\,. \tag{5.263}$$

That means, in turn, that the neutral axis will coincide with the centroidal axis, and that (see Eq. (5.237))

$$\int_A \Sigma_{11} dS = 0\,, \tag{5.264}$$

so that (see Eq. (5.239)) the symmetry condition

$$[\mu_{131}]_{\mathrm{z}=c} = [\mu_{131}]_{\mathrm{z}=-c} \tag{5.265}$$

applies. Note that the function for Σ_{11} given in Eq. (5.169) is compatible with the condition (5.264). Because of Eq. (5.263), and with N_R given in Eq. (5.209), we see that the global equilibrium equation (5.232) is satisfied as well.

G) Stress distribution

Since we are dealing with classical Cauchy stress components Σ_{ij} , the main goal is to calculate the distributions of these components. For the Euler - Bernoulli beam theory we have established, we will now prove, that only parts of the distributions of the internal stresses μ_{ijk} have to be known in order to determine the distributions of the components Σ_{11} , Σ_{13} and Σ_{33} .

Suppose that q(x) and the boundary conditions in Eqs. (5.214) - (5.219) are prescribed, where Q, M and m are given in Eqs. (5.257) - (5.259) as functions of the derivatives of w. By solving Eq. (5.260) we can determine w(x) and then, by substituting into Eqs. (5.257) - (5.259), we can determine the section forces Q(x), M(x) and m(x). Having these functions available, the component Σ_{11} can be established from Eq. (5.169):

$$\Sigma_{11} = -E\mathrm{w}''\mathrm{z} + l^2 E\mathrm{w}''''\mathrm{z}\,. \tag{5.266}$$

In order to compare the structure of this formula with the classical one in Eq. (5.96), we rewrite Σ_{11} as

$$\Sigma_{11} = -\frac{(EI + l^2EA)\mathrm{w}''}{I}\mathrm{z} + \frac{l^2EI\mathrm{w}''''}{I}\mathrm{z} + \frac{l^2EA\mathrm{w}''}{I}\mathrm{z}\,, \tag{5.267}$$

and by invoking Eq. (5.258), we arrive at

$$\Sigma_{11} = \frac{M}{I}\mathrm{z} + \frac{l^2EA\mathrm{w}''}{I}\mathrm{z}\,. \tag{5.268}$$

It can be seen that there is one more term in the formula (5.268), including the axial rigidity EA. Towards deriving a formula for Σ_{13} , we first take the derivative of Eq. (5.268) with respect to x,

$$\partial_{\mathrm{x}}\Sigma_{11} = \frac{M'}{I}\mathrm{z} + \frac{l^2EA\mathrm{w}'''}{I}\mathrm{z}\,, \tag{5.269}$$

and then we make use of Eq. (5.261) to obtain

$$\partial_x \Sigma_{11} = \frac{Q + l^2 EA\mathrm{w}'''}{I} \mathrm{z} \,. \tag{5.270}$$

We introduce this in the equilibrium equation (5.136) to produce

$$\partial_z \Sigma_{13} = -\frac{Q + l^2 EA\mathrm{w}'''}{I} \mathrm{z} \,, \tag{5.271}$$

and integrating from $\hat{\mathrm{z}} = \mathrm{z}$ to $\hat{\mathrm{z}} = c$ (cf. Fig. 16),

$$\int_{\mathrm{z}}^{c} \partial_{\hat{\mathrm{z}}} \Sigma_{13} d\,\hat{\mathrm{z}} = -\frac{Q + l^2 EA\mathrm{w}'''}{I} \int_{\mathrm{z}}^{c} \hat{\mathrm{z}}\, d\hat{z} \,, \tag{5.272}$$

which gives

$$[\Sigma_{13}]_{\hat{\mathrm{z}}=c} - [\Sigma_{13}]_{\hat{\mathrm{z}}=\mathrm{z}} = -\frac{Q + l^2 EA\mathrm{w}'''}{2I} (c^2 - \mathrm{z}^2) \,. \tag{5.273}$$

On appealing to Eq. (5.227),

$$-l^2 E\mathrm{w}''' - [\Sigma_{13}]_{\hat{\mathrm{z}}=\mathrm{z}} = -\frac{\mathrm{Q} + l^2 EA\mathrm{w}'''}{2I} (c^2 - \mathrm{z}^2) \,, \tag{5.274}$$

or

$$\Sigma_{13} = \frac{Q + l^2 EA\mathrm{w}'''}{2I} (c^2 - \mathrm{z}^2) - l^2 E\mathrm{w}''' \,. \tag{5.275}$$

By comparison, we recognize that this distribution of Σ_{13} differs from the classical one in Eq. (5.95) in the term in front of $(c^2 - \mathrm{z}^2)$. Moreover, the term $-l^2 E\mathrm{w}'''$ is absent in Eq. (5.95). Opposite to the classical case, a classical non - vanishing shear stress Σ_{13} now exists on the boundary planes $\mathrm{z} = \pm c$. However, there also exists a non - vanishing, non - classical shear stress part $-\partial_1 \mu_{311} = l^2 E\mathrm{w}'''$ (cf. Eqs (5.222), (5.225)), so that the classical shear traction $[P_1]_{\mathrm{z}=\pm c}$ vanishes (see conditions (5.225), (5.226)).

For deriving the distribution of Σ_{33} we recall from the equilibrium equation (5.137), that

$$\partial_z \Sigma_{33} = -\partial_x \Sigma_{13} \,, \tag{5.276}$$

which gives, after substitution of the result (5.275),

$$\partial_z \Sigma_{33} = -\frac{Q' + l^2 EA\mathrm{w}''''}{2I} (c^2 - \mathrm{z}^2) + l^2 E\mathrm{w}'''' \,. \tag{5.277}$$

Integration from $\hat{\mathrm{z}} = \mathrm{z}$ to $\hat{\mathrm{z}} = c$, keeping in mind condition (5.225) and Eq. (5.262), furnishes

$$\Sigma_{33}(\mathrm{x}, \mathrm{z}) = \frac{q - l^2 EA\mathrm{w}''''}{2I} \left(c^2 \mathrm{z} - \frac{\mathrm{z}^3}{3} - \frac{2}{3} c^3 \right) + l^2 E\mathrm{w}''''(\mathrm{z} - c) \,. \tag{5.278}$$

Similar to the shear stresses, this distribution differs from the classical one in Eq. (5.103) in the factor in front of the term $\left(c^2 \mathrm{z} - \frac{\mathrm{z}^3}{3} - \frac{2}{3} c^3 \right)$ and in the last term on the right - hand side of Eq. (5.278).

Before closing this part, it is of interest to discuss some distributions of the non - classical tractions on the boundaries. Vanishing non - classical traction vectors **R** have been postulated on the boundaries A③, A④, A⑤ and A⑥ (cf. part D) of this section), in accordance with the remarks in Section 4.1.2. Similarly, we shall postulate the non - classical section force m, which is expressed in terms of the component μ_{111}, to vanish on the bounding cross sections A①, A②, i.e.,

$$m_L = m_0 = 0\,. \tag{5.279}$$

It is also worth noticing that, by comparison of Eq. (5.258) with Eq. (5.206),

$$\int_A \partial_{\mathrm{z}}(\mu_{131}\mathrm{z})dS = l^2 EA\mathrm{w}''\,, \tag{5.280}$$

and, on account of Eq. (5.265), that

$$[\mu_{131}]_{\mathrm{z}=\pm c} = l^2 E\mathrm{w}''\,. \tag{5.281}$$

The result (5.280), and comparison of Eq. (5.261) with Eq. (5.246), lead to

$$\int_A \partial_{\mathrm{z}}\mu_{133} dS = 0 \;\Leftrightarrow\; [\mu_{133}]_{\mathrm{z}=c} = [\mu_{133}]_{\mathrm{z}=-c}\,. \tag{5.282}$$

The latter is a symmetry condition stating that the values of μ_{133}(x,z) at the upper and the lower bounding planes are equal to each other for every x. However, it is not necessary to know these whole distributions in order to determine the flexure curve w(x). A similar symmetry condition also applies for the values of μ_{131} in Eq. (5.281). Further, it follows from Eqs. (5.282), (5.193) that

$$Q = Q(\mathrm{x}) = \int_A \Sigma_{13} dS\,. \tag{5.283}$$

G) Discussion

One could agree, that a true one - dimensional formulation should ignore all lateral dimensions and, hence, it should deal with only the dimension variable x. Accordingly, the starting point should be the choice of $W^{(i)}$ as in Eq. (5.82) for the classical case. In the case of the KG - Model, then, Eq. (5.82) could be generalized as follows. The kinematical variable is considered to be w(x) and for classical elasticity, $W^{(i)}$, in Eq. (5.82), contains a quadratic term with respect to w″, which corresponds to the strain $\boldsymbol{\epsilon}$. In a pure one - dimensional approach there exist only derivatives with respect to x, and so w‴ is considered to correspond to the gradient of $\boldsymbol{\epsilon}$. Thus, one can think the appropriate generalization of $W^{(i)}$ to arise by assuming, for the integral in Eq. (5.82), an additional term, which is quadratic with respect to w‴. That means, one might set

$$W^{(i)} = \frac{1}{2}\int_0^L EI\,((\mathrm{w}'')^2 + l^2(\mathrm{w}''')^2)\,d\mathrm{x}\,, \tag{5.284}$$

as the appropriate form modelling an Euler - Bernoulli beam reflecting the KG - Model. Naturally, the internal material length l in Eq. (5.284) should be the same as for the KG - Model in Eq. (3.29). The ansatz (5.284) corresponds to the one - dimensional gradient elasticity beam theory adopted in Papargyri - Beskou et al. [30], whenever their surface energy term is omitted, i.e., whenever their material parameter l (not to be confused with l in the present thesis) is vanishing. It is, however, emphasized, that the theory in [30] has not been introduced as a generalization of the KG - Model. The further steps for the ansatz in Eq. (5.284) are quite similar to those in part E) of this section. The final results can be obtained by setting A = 0 in the equations of part E). It is clear that, for A ≠ 0 the two one - dimensional approaches are different.

Note also that the form of the free energy in Eq. (5.171) has been adopted for the first time in [23] in a context dealing only with section forces. In difference to this work, the approach pursued here allows to elaborate the relations between section forces and stress components.

5.4.1 Solution for specific boundary conditions

For the sake of comparison, we seek solutions for the problem in Fig. 17, for the case of the KG - Model. That means, we have to solve for

$$q(\mathrm{x}) = q_0 = \text{const.} \tag{5.285}$$

the differential equation (5.260), which in dimensionless form reads

$$\left[1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right]\tilde{\mathrm{w}}'''' - \tilde{l}^2\tilde{\mathrm{w}}'''''' = \frac{3\tilde{q}_0}{4\tilde{b}\tilde{c}^3}\,. \tag{5.286}$$

As above, A = 4 b c, the notation introduced in Eq. (5.123) applies, and in addition

$$\tilde{l} \coloneqq \frac{l}{L}. \tag{5.287}$$

In accordance with Eqs. (5.122) and (5.279), the boundary conditions are specified as

$$\mathrm{w}'(0) = 0\,,\ \ \mathrm{w}(0) = 0\,,\ \ M_L = 0\,,\ \ Q_L = 0\,,\ \ m_L = m_0 = 0\,, \tag{5.288}$$

or equivalently (cf. Eqs. (5.257) - (5.259)),

$$\begin{gathered} \mathrm{w}'(0) = 0,\ \ \mathrm{w}(0) = 0\,,\ \ -(EI + l^2EA)\mathrm{w}''(L) + l^2EI\mathrm{w}''''(L) = 0\,, \\ -(EI + l^2EA)\mathrm{w}'''(L) + l^2EI\mathrm{w}'''''(L) = 0\,,\ \mathrm{w}'''(L) = 0\,,\ \mathrm{w}'''(0) = 0\,. \end{gathered} \tag{5.289}$$

They can be recast in the dimensionless forms

$$\begin{gathered} \tilde{\mathrm{w}}'(0) = 0\,,\ \ \tilde{\mathrm{w}}(0) = 0\,,\ \ \left[1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right]\tilde{\mathrm{w}}''(1) - \tilde{l}^2\tilde{\mathrm{w}}''''(1) = 0\,, \\ \left[1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right]\tilde{\mathrm{w}}'''(1) - \tilde{l}^2\tilde{\mathrm{w}}'''''(1) = 0\,,\ \ \tilde{\mathrm{w}}'''(1) = 0\,,\ \ \tilde{\mathrm{w}}'''(0) = 0\,. \end{gathered} \tag{5.290}$$

The solution of the differential Eq. (5.286) reads

$$\begin{aligned} \tilde{\mathrm{w}} = {} & \tilde{\mathrm{c}}_1 + \tilde{\mathrm{c}}_2\tilde{\mathrm{x}} + \tilde{\mathrm{c}}_3\tilde{\mathrm{x}}^2 + \tilde{\mathrm{c}}_4\tilde{\mathrm{x}}^3 + \tilde{\mathrm{c}}_5\mathrm{e}^{\sqrt{\frac{1+3\tilde{l}^2/\tilde{c}^2}{\tilde{l}^2}}\,\tilde{\mathrm{x}}} + \tilde{\mathrm{c}}_6\mathrm{e}^{-\sqrt{\frac{1+3\tilde{l}^2/\tilde{c}^2}{\tilde{l}^2}}\,\tilde{\mathrm{x}}} \\ & + \frac{1}{24\left(1 + 3\tilde{l}^2/\tilde{c}^2\right)}\frac{3\tilde{q}_0}{4\tilde{b}\tilde{c}^3}\tilde{\mathrm{x}}^4\,, \end{aligned} \tag{5.291}$$

where the constants of integration $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \tilde{c}_5, \tilde{c}_6$ can be calculated by evaluating boundary conditions (5.290) (see Appendix B). From Eq. (5.258) and the definitions in Eqs. (5.123), (5.287),

$$\widetilde{M} = -\left[1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right]\widetilde{\mathrm{w}}'' + \tilde{l}^2\widetilde{\mathrm{w}}'''' \,, \tag{5.292}$$

and from Eq. (5.268), we conclude that

$$\tilde{\Sigma}_{11} \coloneqq \frac{\Sigma_{11}}{E} = -\widetilde{\mathrm{w}}''\tilde{\mathrm{z}} + \tilde{l}^2\widetilde{\mathrm{w}}''''\tilde{\mathrm{z}} \,, \tag{5.293}$$

$$\tilde{\Sigma}_{11}(\tilde{\mathrm{x}} = 0, \tilde{\mathrm{z}}) = -\widetilde{\mathrm{w}}''(0)\tilde{\mathrm{z}} + \tilde{l}^2\widetilde{\mathrm{w}}''''(0)\tilde{z} \,, \tag{5.294}$$

$$\left(\tilde{\Sigma}_{11}\right)_{\max} \coloneqq \max\{\left|\tilde{\Sigma}_{11}(0, \tilde{\mathrm{z}})\right|\} = \left|\tilde{\Sigma}_{11}(0, \tilde{\mathrm{z}} = \pm\tilde{c})\right| . \tag{5.295}$$

Using the definitions in Eq. (5.123), we can recast Eqs. (5.275), (5.278) in dimensionless forms,

$$\tilde{\Sigma}_{13} \coloneqq \frac{\Sigma_{13}}{E} = \left[-\frac{1}{2}\widetilde{\mathrm{w}}''' + \frac{1}{2}\tilde{l}^2\widetilde{\mathrm{w}}'''''\right](\tilde{\mathrm{c}}^2 - \tilde{\mathrm{z}}^2) - \tilde{l}^2\widetilde{\mathrm{w}}''', \tag{5.296}$$

$$\tilde{\Sigma}_{33} \coloneqq \frac{\Sigma_{33}}{E} = \left(\frac{\widetilde{\mathrm{w}}'''' - \tilde{l}^2\widetilde{\mathrm{w}}''''''}{2}\right)\left(\tilde{\mathrm{c}}^2\tilde{\mathrm{z}} - \frac{\tilde{\mathrm{z}}^3}{3} - \frac{2}{3}\tilde{\mathrm{c}}^3\right) - \tilde{l}^2\widetilde{\mathrm{w}}''''(\tilde{\mathrm{c}} - \tilde{\mathrm{z}}) \,. \tag{5.297}$$

5.5 Comparison between the responses predicted by the KG - Model and classical elasticity

For the cantilever beam in Fig. 17, we are going to present distributions of the deflection curve $\tilde{\mathrm{w}}$ and of the derivatives $\tilde{\mathrm{w}}'$, $\tilde{\mathrm{w}}''$, $\tilde{\mathrm{w}}'''$ along the $\tilde{\mathrm{x}}$ - axis. As seen in Fig. 18 a), the deflections predicted by the KG - Model become smaller with increasing values of $\tilde{l}$ and are always below the one predicted by the classical elasticity model, indicating gradient stiffening effect. Generally, whenever $\tilde{l} \to 0$, there is no uniform convergence of the responses predicted by the KG - Model to the corresponding responses predicted by the classical elasticity model. This happens because the governing differential equation for the deflection in the case of classical elasticity is of fourth - order (cf. Eq. (5.73)), whereas for the case of the KG - Model, it is of sixth - order (cf. Eq. (5.260)). In the latter case, two additional boundary conditions have to be accounted for, in comparison to the classical case, indicating singular perturbation relationships (see Broese et al. [7]). However, for the special boundary conditions $m_L = m_0 = 0$ in Eq. (5.288), it can be recognized from figures 18 a), b), c), that the distributions of $\tilde{\mathrm{w}}$, $\tilde{\mathrm{w}}'$, $\tilde{\mathrm{w}}''$ converge uniformly to the corresponding distributions predicted by the classical elasticity. Concerning the distributions of $\tilde{\mathrm{w}}'''$, we have pointwise convergence

$$\lim_{\tilde{l}\to 0}[\tilde{\mathrm{w}}'''(\tilde{\mathrm{x}})] = [\tilde{\mathrm{w}}'''(\tilde{\mathrm{x}})]_{\mathrm{classical}} \ \text{ for } \ \tilde{\mathrm{x}} \in (0,1] \,, \tag{5.298}$$

whereas at $\tilde{\mathrm{x}} = 0$, the limit function $\lim\limits_{\tilde{l}\to 0}[\tilde{\mathrm{w}}'''(\tilde{\mathrm{x}})]$ is not continuous (see Fig. 18 d). Since $\tilde{\Sigma}_{13}$ and $\tilde{\Sigma}_{33}$ depend on $\tilde{\mathrm{w}}'''$ (see Eqs. (5.296), (5.297)). One would expect that this behaviour carries over to the distributions of $\tilde{\Sigma}_{13}$ and $\tilde{\Sigma}_{33}$. However, this is not the case as we will see below.

Furthermore, Fig. 19 illustrates the effect of the section geometry on the predicted responses for both the classical and gradient elasticity. For constant width $\tilde{b}$ and various heights $\tilde{c}$, several distributions of $\tilde{\mathrm{w}}$ are displayed in this figure. Besides quantitative differences, the same trend can be recognized for both models, as one would expect, i.e., decreasing values of $\tilde{\mathrm{w}}$ with increasing values of $\tilde{c}$. However, there is an important difference between the two models in what concerns the effect of the section height c, which can be demonstrated by considering the ratio $\frac{M}{2c}$ for $c \ \to 0$. From Eqs. (5.94), (5.258) and the definition (5.71), we have:

$$\text{Classical case:} \ \ \lim_{\mathrm{c}\to 0}\frac{M}{2c} = \lim_{\mathrm{c}\to 0}\left(-\frac{2}{3}Ebc^2\mathrm{w}''\right) = 0 \,, \tag{5.299}$$

whereas

$$\begin{aligned} \text{KG - Model:} \ \ \lim_{\mathrm{c}\to 0}\frac{M}{2c} &= \lim_{\mathrm{c}\to 0}\left[-E\left(\frac{2}{3}bc^2 + l^2 2b\right)\mathrm{w}'' + \frac{2}{3}l^2 Ebc^2\mathrm{w}''''\right] \\ &= -El^2 2b\mathrm{w}'' \,. \end{aligned} \tag{5.300}$$

The interpretation of the above is that, if we set the height of the section close to zero, for $\frac{M}{2c}$ the classical case predicts properties of membrane. On the other hand, for $\frac{M}{2c}$, the KG - Model retains some stiffness.

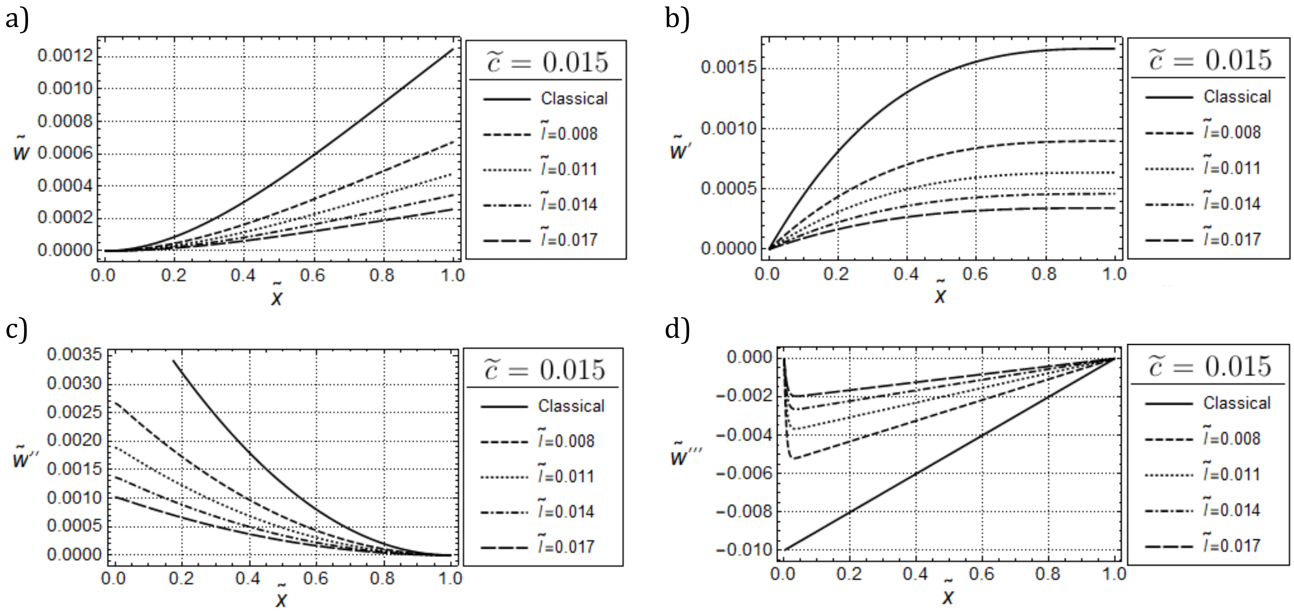


Figure 18: Distribution of a) flexure $\tilde{w}$, b) first derivative of $\tilde{w}$, c) second derivative of $\tilde{w}$ and d) third derivative of $\tilde{w}$ for the cantilever beam with $\tilde{c} = 0.015$ for both the classical case and the KG – Model and for various values of $\tilde{l}$.

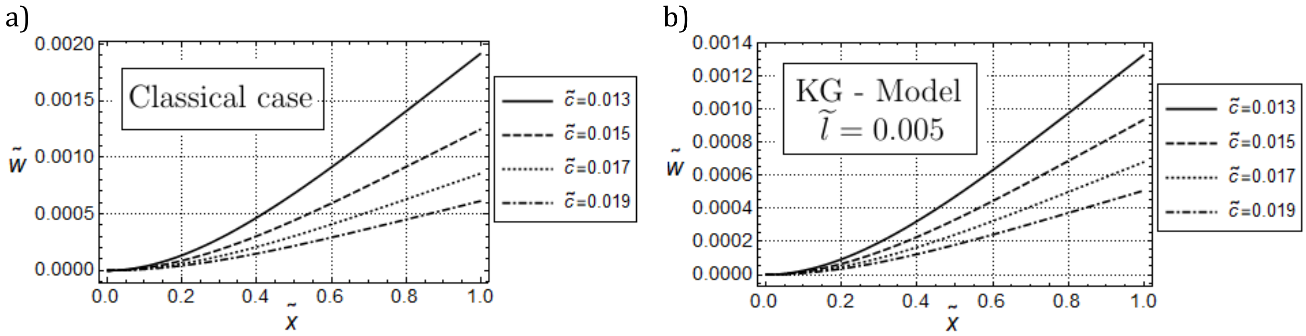


Figure 19: Distribution of flexure $\tilde{w}$ for the cantilever beam for various values of $\tilde{c}$ for a) classical case and b) KG - Model, keeping $\tilde{l}$ constant.

Next, we will present distributions of normal stresses $\tilde{\sigma}_x$ and $\tilde{\Sigma}_{11}$. For $\tilde{x} = 0$, Fig. 20 shows that $\tilde{\sigma}_x$, $\tilde{\Sigma}_{11}$ are point symmetric with respect to the origin of the coordinate system. The amount of the stresses predicted by the KG - Model is smaller than the classical case and decreases with increasing values of $\tilde{l}$. Obviously, the stresses predicted by the KG - Model converge uniformly to the stress predicted by the classical elasticity,

$$\lim_{\tilde{l} \to 0} \tilde{\Sigma}_{11} = \tilde{\sigma}_x \;\text{ for }\; \tilde{z} \in [-\tilde{c}, \tilde{c}] \;\text{ and }\; \tilde{x} = 0\,. \tag{5.301}$$

Since the maximum amount of $\tilde{\sigma}_x$, $\tilde{\Sigma}_{11}$ is attained for $\tilde{z} = \pm\tilde{c}$, in order to compare $\tilde{x}$ - distributions of these stresses, it suffices to focus on $\tilde{z} = -\tilde{c}$. Similar to the $\tilde{z}$ - distributions, the $\tilde{x}$ - distributions in Fig. 21 suggest decreasing amount of stresses predicted by the KG - Model with increasing values of $\tilde{l}$, at every $\tilde{x}$, and that these distributions converge uniformly to the one predicted by classical elasticity,

$$\lim_{\tilde{l} \to 0} \tilde{\Sigma}_{11} = \tilde{\sigma}_x \;\text{ for }\; \tilde{x} \in [0,1] \;\text{ and }\; \tilde{z} = -c\,. \tag{5.302}$$

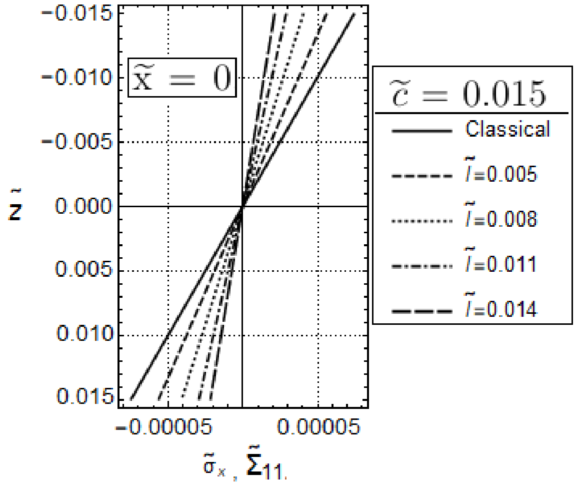


Figure 20: Distributions of stress $\tilde{\sigma}_x$ for the classical case and $\tilde{\Sigma}_{11}$ for the KG - Model along the $\tilde{z}$ - axis of the cantilever beam when $\tilde{x} = 0$.

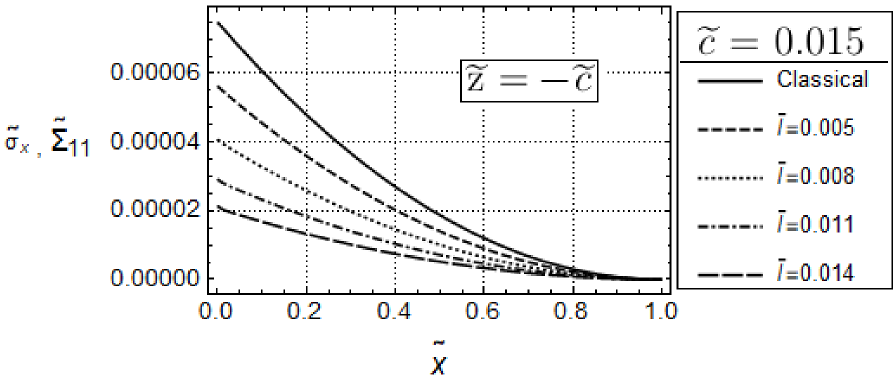


Figure 21: Distributions of stress $\tilde{\sigma}_x$ for the classical case and $\tilde{\Sigma}_{11}$ for the KG - Model along the $\tilde{x}$ - axis of the cantilever beam when $\tilde{z} = -\tilde{c}$.

In the following, distributions of shear stresses $\tilde{\tau}$, $\tilde{\Sigma}_{13}$ are presented. First, Fig. 22 shows the distributions along the $\tilde{z}$ - axis. At $\tilde{x} = 0$ (see Fig. 22 a)) all curves of $\tilde{\Sigma}_{13}$ coincide with the classical one irrespective of the $\tilde{l}$ values. Moreover, as noted after Eq. (5.275), the shear stress $\tilde{\Sigma}_{13}$ does not vanish on the boundary planes, opposite to the classical shear stress. At every other point of $\tilde{x}$ along the beam ($\tilde{x} > 0$), the distributions in Fig. 22 b) hold. Here, it can be seen, that the stress distributions predicted by the KG - Model converge uniformly to the one predicted by the classical elasticity. Moreover, as noted in Eq. (5.275), the shear stress $\tilde{\Sigma}_{13}$ for the KG - Model does not vanish on the boundary planes $\tilde{z} = \pm\tilde{c}$ as opposed to the classical shear stress $\tilde{\tau}$. Additionally, KG - Model predicts smaller stresses near the centroidal axis and greater stresses near the upper and lower boundary planes than the classical elasticity. Note, also, that the stresses $\tilde{\tau}$, $\tilde{\Sigma}_{13}$ are very small compared to the normal stresses $\tilde{\sigma}_x$, $\tilde{\Sigma}_{11}$. This is a consequence of the Euler - Bernoulli beam theory for slender beams. From experimental data, we know that this hypothesis holds for beams with small sections compared to the total length. Secondly, Fig. 23 shows stress distributions along the $\tilde{x}$ - axis, at $\tilde{z} = 0$, where $\tilde{\tau}$, $\tilde{\Sigma}_{13}$ reach their maximum values. It can be seen that the KG - Model predicts smaller stresses at every $\tilde{x} > 0$.

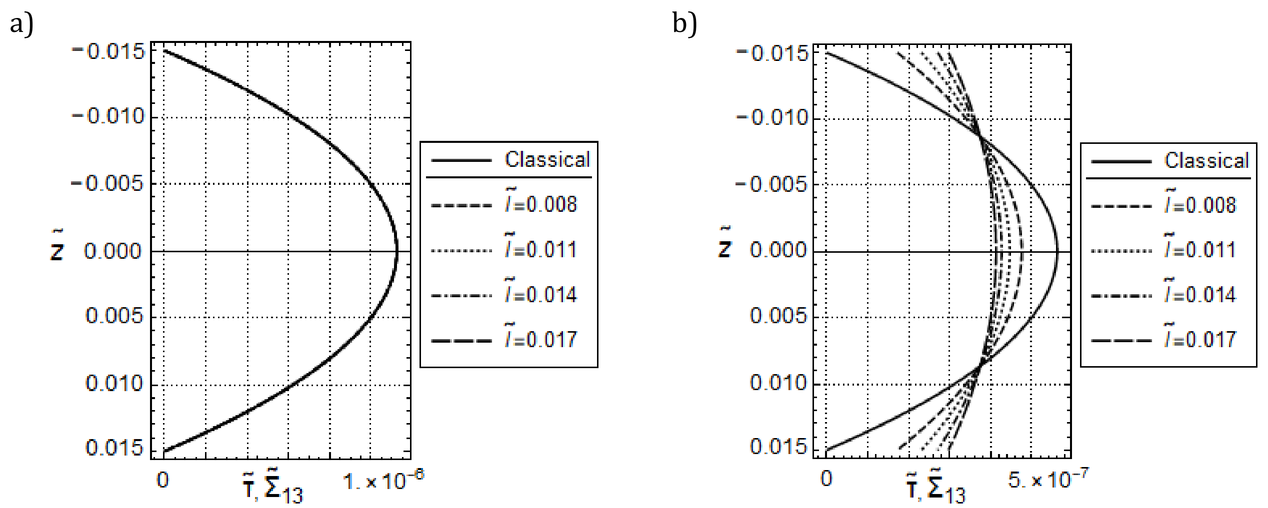


Figure 22: Distribution of shear stresses for the cantilever beam along the $\tilde{z}$ - axis at a) left fixed end and b) middle of the beam for both the classical case and the KG - Model.

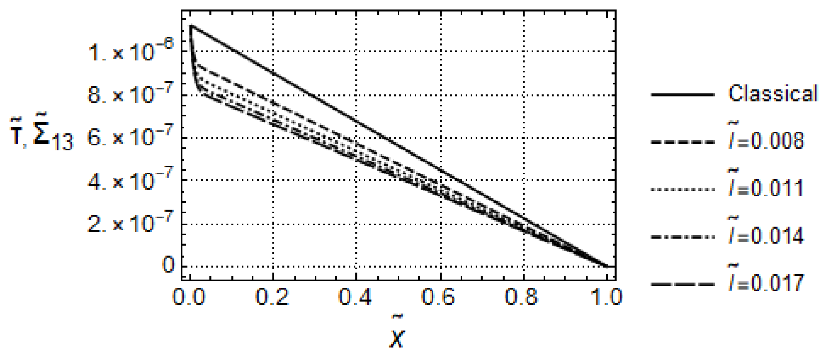


Figure 23: Distribution of shear stresses for the cantilever beam along the $\tilde{x}$ - axis, when $\tilde{z} = 0$, for both the classical case and the KG - Model.

Finally, distributions of normal stresses $\tilde{\sigma}_z$, $\tilde{\Sigma}_{33}$ along the cross section of the beam are presented. As seen in the Fig. 24 a), at the left end of the beam ($\tilde{x} = 0$), there are significant differences between the responses predicted by the KG - Model and those predicted by classical elasticity. On one hand, the amount of the stresses is higher in the case of KG - Model almost everywhere. On the other hand, KG - Model predicts negative stresses above the neutral axis ($-\tilde{c} < \tilde{z} < 0$) and mainly positive stresses below the neutral axis ($0 < \tilde{z} < \tilde{c}$), while the classical elasticity model predicts negative stresses everywhere along the $\tilde{z}$ - axis. In other words, for the KG - Model, if we consider a particle of the beam above the neutral axis, it is subject to compression and if we consider a particle below the neutral axis, it is subject to tension (see Fig. 25). This form of $\tilde{\Sigma}_{33}$ prevails in a very small boundary layer. After that, at any $\tilde{x}$ = const. , the shapes of $\tilde{\Sigma}_{33}$ – curves in Fig. 24 b) hold, i.e., they are similar to the classical one. For every $\tilde{x}$, however, the $\tilde{\Sigma}_{33}$ – distributions converge uniformly to the ones according to classical elasticity. Additionally, on the lower end of the cross section ($\tilde{z} = \tilde{c}$), the stresses $\tilde{\sigma}_z$, $\tilde{\Sigma}_{33}$ vanish and on the upper end ($\tilde{z} = -\tilde{c}$), they depend on the external load $\tilde{q}_0$ applied to the beam. Similar to stresses $\tilde{\tau}$ and $\tilde{\Sigma}_{13}$, stresses $\tilde{\sigma}_z$ and $\tilde{\Sigma}_{33}$ are also very small compared to stresses $\tilde{\Sigma}_{11}$ because of the Euler - Bernoulli beam hypothesis.

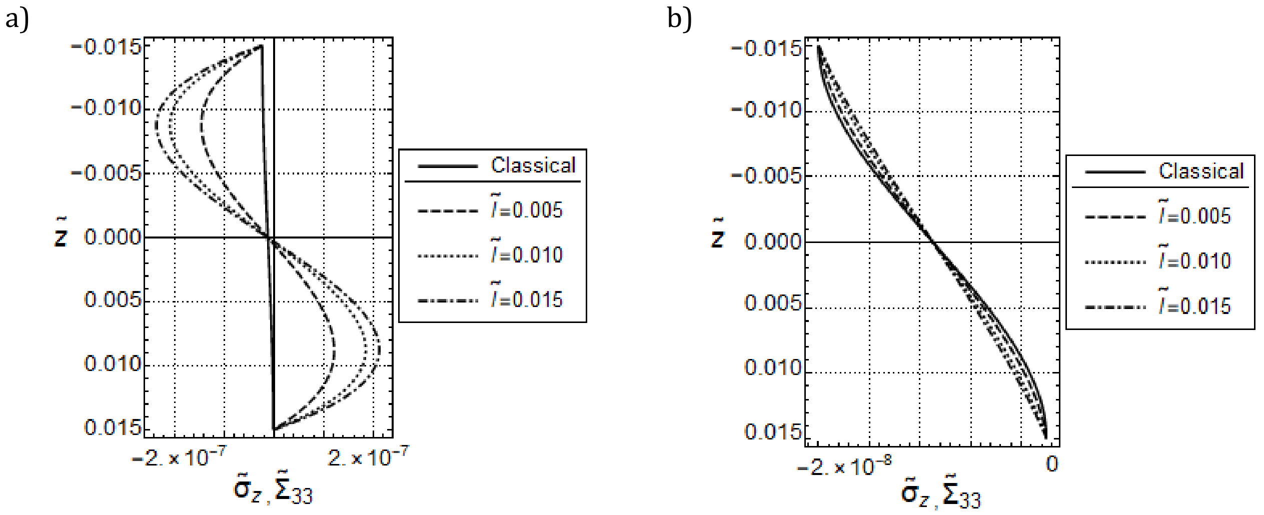


Figure 24: Distribution of normal stresses $\tilde{\sigma}_z$, $\tilde{\Sigma}_{33}$ along the $\tilde{z}$ - axis for the cantilever beam at a) point $\tilde{x} = 0$, b) any other point $\tilde{x}$.

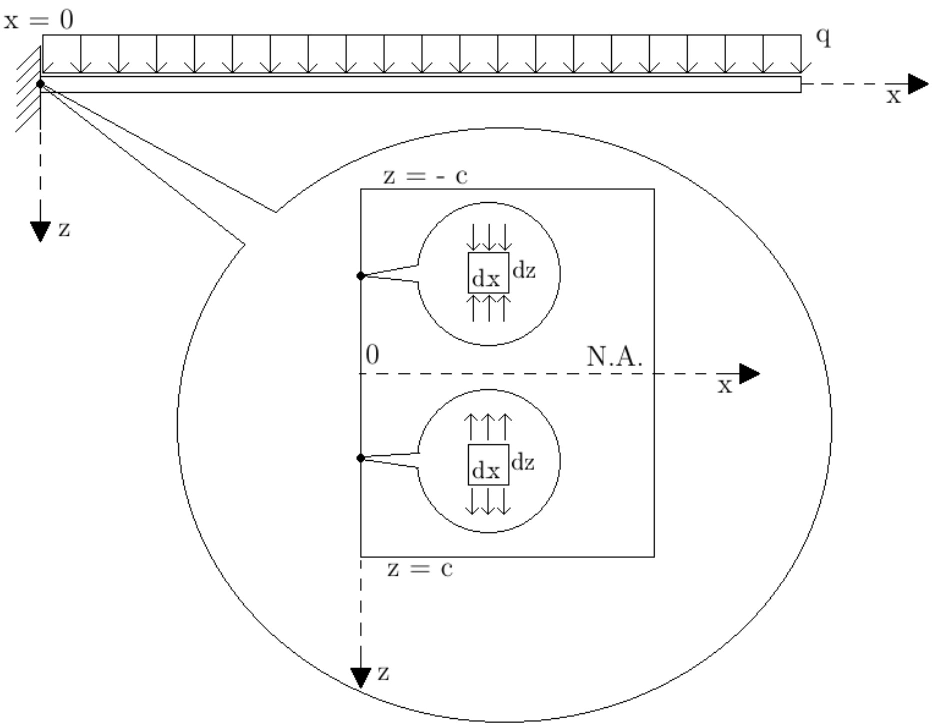


Figure 25: Stress $\tilde{\Sigma}_{33}$ acting on particles at the left end of the beam for the case of the KG - Model. Above the neutral axis, we have a case of compression (negative stresses) and below the neutral axis we have a case of tension (positive stresses).

6 Buckling of Euler - Bernoulli columns

Let, once more, the assumptions made in section 5.1.2 A) hold and suppose the coordinate system $\{x_i\}$ to be a centroidal one. The developed Euler - Bernoulli theory is linear and therefore it furnishes unique solutions. The common way to work around this problem is to introduce some non - linearity in the theory, e.g., by taking into account the axial deformation in the deflected configuration. To be more specific, consider the simply supported beam in Fig. 26, under the axial compressive force F. The applied load is at the beginning zero (State ①) and it is continuously increasing, while the beam remains straight, until a critical load $F = F_{cr}$ is reached (State ②), at which buckling occurs. At this load, the beam might remain further straight (unstable solution), or it might leave the straight configuration and move to a deflected configuration (State ③), the latter being an equilibrium configuration as well. As others stated, instability is noticeable at $F = F_{cr}$ as a bifurcation phenomenon, where for the same external load $F = F_{cr}$, two different configurations are possible. Until State ②, there is only axial displacement $U = U(\mathrm{x})$ due to the compressive force. Going from state State ② to State ③, the fundamental assumption made is that U remains constant, for the axial force is constant. However, there are two further contributions to the total axial displacement $\mathrm{u} = \mathrm{u}_1$. The first one is $-\mathrm{w}'\mathrm{z}$ due to bending, as in the linear theory (cf. Eq. (5.26)). The second is related to the displacement $\overline{\Delta L}$ in Fig. 26; it is also due to bending and describes the contribution in axial displacement due to the buckled configuration compared with the straight configuration. In linear theory such deformations are disregarded. We can calculate this contribution with the help of Fig. 27, which shows a differential element of the beam, with length $d\mathrm{x}$, in the straight and the buckled configurations. It is readily verified, that

$$\zeta(\mathrm{x}) = d\mathrm{x} - d\mathrm{x}\cos\mathrm{w}' = d\mathrm{x} - d\mathrm{x}\{1 - \frac{1}{2}(\mathrm{w}')^2 + \cdots.\} \approx \frac{1}{2}(\mathrm{w}')^2\,, \tag{6.1}$$

so that the total displacement u becomes

$$\mathrm{u} = \mathrm{u}(\mathrm{x},\mathrm{z}) = U(\mathrm{x}) - \mathrm{w}'\mathrm{z} + \int_0^{\mathrm{x}} \frac{1}{2}\big(\mathrm{w}'(\hat{\mathrm{x}})\big)^2 d\hat{\mathrm{x}}\,. \tag{6.2}$$

The u_3 component is assumed to be the same as in the linear theory (cf. Eq. (5.26)). Altogether

$$\mathbf{u} = \mathbf{u}(\mathrm{x},\mathrm{z}) \mathrel{\widehat{=}} \begin{pmatrix} U^0(x) - \mathrm{w}'(\mathrm{x})\mathrm{z} \\ 0 \\ \mathrm{w}(\mathrm{x}) \end{pmatrix}, \tag{6.3}$$

where

$$U^0(\mathrm{x}) = U(\mathrm{x}) + \int_0^{\mathrm{x}} \frac{1}{2}\big(\mathrm{w}'(\hat{\mathrm{x}})\big)^2 d\hat{\mathrm{x}}\,. \tag{6.4}$$

Accordingly,

$$\epsilon_{ij} = \frac{1}{2}\big(\partial_i \mathrm{u}_j + \partial_j \mathrm{u}_i\big) \mathrel{\widehat{=}} \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

with

$$\epsilon_{11} = \epsilon_{11}^0 - \mathrm{w}''\mathrm{z}\,, \tag{6.6}$$

$$\epsilon_{11}^0 = U' + \frac{1}{2}(\mathrm{w}')^2\,. \tag{6.7}$$

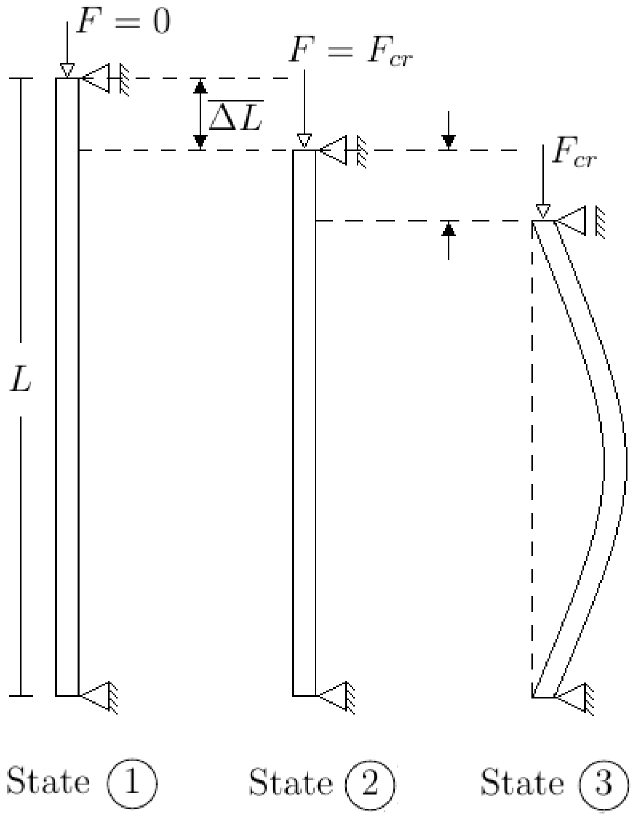


Figure 26: Simply supported beam under axial compressive force F.

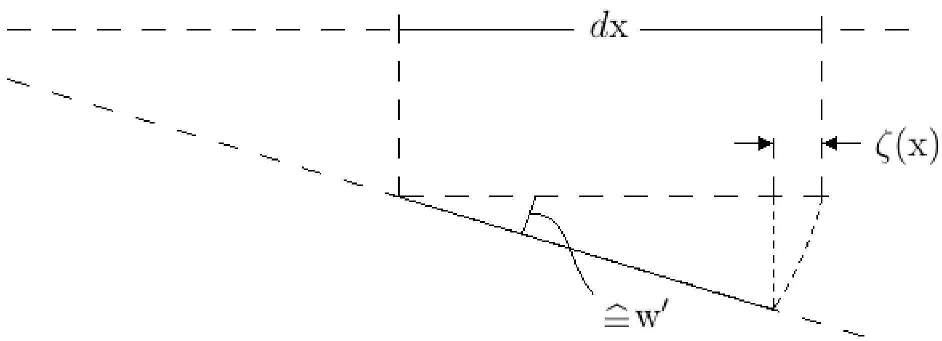


Figure 27: Differential element with length dx in the straight and the buckled configurations.

Assuming dead - loading external forces to apply, the principle of stationary (total) potential energy asserts that, for the beam $\mathfrak{B}$ and for every sub - body $\Delta\mathfrak{B}$ of it, the necessary and sufficient condition for equilibrium is

$$\delta\Pi = 0\,, \tag{6.8}$$

$$\Pi \coloneqq \Pi^{(i)} + \Pi^{(e)}\,, \tag{6.9}$$

for all admissible variations of U and w. In these equations, $\Pi^{(i)}$ and $\Pi^{(e)}$ are the potentials of the internal and the external forces, respectively. With

$$\Pi^{(i)} \equiv W^{(i)} = \int_V \psi \, dV \,, \tag{6.10}$$

we can verify from Eqs. (5.170) and (6.6), that

$$\Pi^{(i)} = \Pi^{(i)}(U, \mathrm{w}) = \int_V \frac{1}{2} E[\epsilon_{11}^2 + l^2(\partial_1\epsilon_{11})^2 + l^2(\partial_3\epsilon_{11})^2]dV \;\Rightarrow \tag{6.11}$$

$$\delta\Pi^{(i)} = E\int_0^L \int_A [\epsilon_{11}\delta\epsilon_{11} + l^2(\partial_1\epsilon_{11}) + l^2\mathrm{w}''\delta\mathrm{w}'']dS\,d\mathrm{x}\,. \tag{6.12}$$

After lengthy manipulations, using Eqs. (6.6), (6.7) and steps similar to those in section 5, we arrive at

$$\delta\Pi^{(i)} = [f_1\delta U]_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + [f_2\delta U']_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + [(f_1\mathrm{w}' + f_3')\delta\mathrm{w}]_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + [(-f_2\mathrm{w}' + f_3)\delta(-\mathrm{w}')]_{\mathrm{x}_1=0}^{\mathrm{x}_1=L}$$

$$+[f_4\delta(-\mathrm{w}'')]_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + \int_0^L -f_1'\delta U d\mathrm{x}_1 + \int_0^L [-f_3'' - (f_1\mathrm{w}')']\delta\mathrm{w}\,d\mathrm{x}_1\,, \tag{6.13}$$

where

$$f_1 \coloneqq EA[\epsilon_{11}^0 - l^2(\partial_1\partial_1\epsilon_{11}^0)]\,, \qquad f_2 \coloneqq l^2EA\partial_1\epsilon_{11}^0\,, \tag{6.14}$$

$$f_3 \coloneqq -EI(\mathrm{w}'' - l^2\mathrm{w}'''') - l^2EA\mathrm{w}''\,, \qquad f_4 = -l^2EI\mathrm{w}'''\,. \tag{6.15}$$

In analogy to Simitses [31], section 3.2, we set for the variation in the potential of the external forces

$$-\delta\Pi^{(e)} = [N\delta U]_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + [H\delta U']_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + [Q\delta\mathrm{w}]_{\mathrm{x}_1=0}^{\mathrm{x}_1=L} + [M\delta(-\mathrm{w}')]_{\mathrm{x}_1=0}^{\mathrm{x}_1=\mathrm{L}} + [m\delta(-\mathrm{w}'')]_{\mathrm{x}_1=0}^{\mathrm{x}_1=\mathrm{L}}\,. \tag{6.16}$$

For equilibrium, Eq. (6.8) must hold for all admissible variations of U and w, and by using a standard line of argumentation, we conclude, after some rearrangement of terms, that

$$\text{a) } N = f_1\,, \quad H = f_2\,, \quad Q = f_3' + f_1\mathrm{w}', \quad M = f_3 - f_2\mathrm{w}', \quad m = f_4\,, \tag{6.17}$$

$$\text{b) } f_1' = 0\,, \quad -f_3'' - (f_1\mathrm{w}')' = 0\,, \tag{6.18}$$

c) either N or U, either H or U', either Q or w, either M or w$'$ and either m or w$''$ (6.19)

have to be prescribed at $\mathrm{x}_1 = 0, L$.

For the straight bar under the action of a compressive force F, we deal with here, we have

$$N(L) = -F \;\Rightarrow\; N' = 0\,, \qquad N = f_1 = \text{const.} = -F\,. \tag{6.20}$$

If, in addition, we set

$$H(0) = f_2(0) = 0\,, \qquad H(L) = f_2(L) = 0\,, \tag{6.21}$$

then Eqs. (6.17) – (6.21) yield the differential equation

$$-f_3'' - f_1\mathrm{w}'' = 0 \;\Rightarrow \tag{6.22}$$

$$(EI + l^2EA)\mathrm{w}'''' - l^2EI\mathrm{w}'''''' + F\mathrm{w}'' = 0\,, \tag{6.23}$$

which is subject to the boundary conditions

$$\text{either } Q = f_3' - F\mathrm{w}' \;\text{ or }\; \mathrm{w}\,, \tag{6.24}$$

$$\text{either } M = f_3 \;\text{ or }\; \mathrm{w}' \;\text{ and} \tag{6.25}$$

$$\text{either } m = f_4 \;\text{ or }\; \mathrm{w}'' \tag{6.26}$$

have to be prescribed at $\mathrm{x} = 0, L$. From these equations the critical load F_{cr} for buckling can be determined.

Equations (6.22) - (6.26) are comparable with the corresponding equations derived in Papargyri - Beskou et al. [30] and Lazopoulos and Lazopoulos [23]. In particular, they are the same with those in Lazopoulos and Lazopoulos [23], whenever l_{x} in the latter is set equal to zero, provided the boundary conditions (6.21) apply. If, however, the section force H does not satisfy homogenous boundary conditions, then the two approaches are different.

6.1 The eigenvalue problem

For definiteness, we consider the problem of a simply supported beam with the homogenous boundary conditions

$$\mathrm{w}(0) = \mathrm{w}(L) = M(0) = M(L) = m(0) = m(L) = 0\,. \tag{6.27}$$

Use of the dimensionless variables introduced in section 5.3 leads to the dimensionless form of the differential equation (6.23)

$$\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\mathrm{w}}'''' - \tilde{l}^2\tilde{\mathrm{w}}'''''' + \tilde{k}^2\tilde{\mathrm{w}}'' = 0\,, \tag{6.28}$$

and to the dimensionless form of the boundary conditions (6.27)

$$\tilde{\mathrm{w}}(0) = 0\,,\quad \tilde{\mathrm{w}}(1) = 0\,,\quad -\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\mathrm{w}}''(0) + \tilde{l}^2\tilde{\mathrm{w}}''''(0) = 0\,, \tag{6.29}$$

$$-\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\mathrm{w}}''(1) + \tilde{l}^2\tilde{\mathrm{w}}''''(1) = 0\,,\qquad \tilde{\mathrm{w}}'''(0) = \tilde{\mathrm{w}}'''(1) = 0\,, \tag{6.30}$$

where

$$\tilde{k} := \frac{FL^2}{EI}\,. \tag{6.31}$$

With the help of the definitions

$$\tilde{\xi} := \frac{1}{\sqrt{2}\,\tilde{l}}\left\{\left[\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)^2 + 4\tilde{l}^2\tilde{k}^2\right]^{\frac{1}{2}} - \left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\right\}^{\frac{1}{2}}, \tag{6.32}$$

$$\tilde{\theta} := \frac{1}{\sqrt{2}\,\tilde{l}}\left\{\left[\left(1+3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)^2 + 4\tilde{l}^2\tilde{k}^2\right]^{\frac{1}{2}} + \left(1+3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\right\}^{\frac{1}{2}}, \tag{6.33}$$

the general solution of (6.28) reads

$$\tilde{w}(\tilde{x}_1) = \tilde{c}_1 + \tilde{c}_2\tilde{x}_1 + \tilde{c}_3 \sin\tilde{\xi}\,\tilde{x}_1 + \tilde{c}_4 \cos\tilde{\xi}\,\tilde{x}_1 + \tilde{c}_5 \sinh\tilde{\theta}\,\tilde{x}_1 + \tilde{c}_6 \cosh\tilde{\theta}\,\tilde{x}_1\,, \tag{6.34}$$

with $\tilde{c}_1$, ..., $\tilde{c}_6$ being dimensionless constants of integration, which must satisfy the boundary conditions (6.29), (6.30).

This is an eigenvalue problem with eigenvalues $\tilde{k}$, and leads to a homogeneous system of equations for the constants of integration. Non - trivial solutions of this problem exist if and only if

$$\det\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0\,, \tag{6.35}$$

where a_{11} , ..., a_{44} are given in Appendix C.

Condition (6.35) is a non - linear equation for $\tilde{k}$, which may be solved numerically. With regard to Eq. (6.31), the solution $\tilde{k}_{cr}$ yields the critical load $F_{cr}^{(KG)}$ for the KG model,

$$F_{cr}^{(KG)} = \frac{EI}{L^2}\tilde{k}_{cr}\,. \tag{6.36}$$

The corresponding critical Euler load for classical elasticity is (cf., e.g., Simitses [31], p.59)

$$F_{cr}^{(cl.)} = \frac{EI}{L^2}\pi^2\,. \tag{6.37}$$

Solutions $F_{cr}^{(KG)}$, parametrized with the internal material length $\tilde{l}$, are illustrated in Fig. 28. The height of the cross section has been set $2\tilde{c} = 0.01$. It can be recognized that $F_{cr}^{(KG)}/F_{cr}^{(cl.)}$ increases for increasing values of $\tilde{l}$, indicating a gradient stiffening effect for buckling problems.

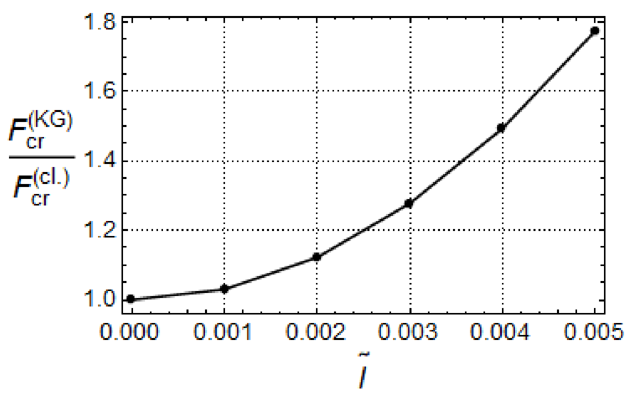


Figure 28: Variation of the ratio $F_{cr}^{(KG)}/F_{cr}^{(cl.)}$.

It is also of interest to illustrate the effect of section geometry on the predicted $F_{cr}^{(KG)}/F_{cr}^{(cl.)}$. For constant width $\tilde{b}$ and keeping $\tilde{l}$ constant ($\tilde{l} = 0.001$), solutions $F_{cr}^{(KG)}/F_{cr}^{(cl.)}$ are displayed in Fig. 29. The differences between the KG - Model and the classical elasticity diminish as the height of the section increases and $F_{cr}^{(KG)}/F_{cr}^{(cl.)} \to 1$ for sufficiently large values of $\tilde{c}$, i.e., distributions predicted by both the KG - Model and the classical elasticity converge.

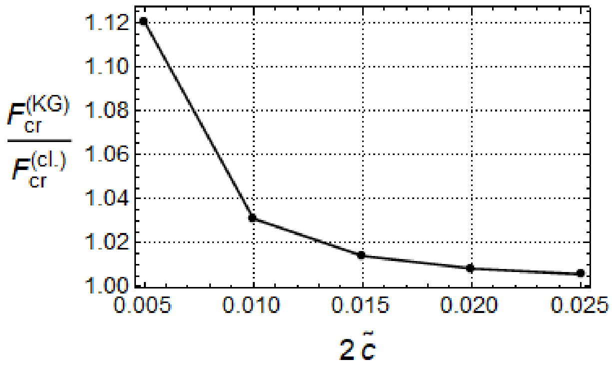


Figure 29: Variation of the ratio $F_{cr}^{(KG)}/F_{cr}^{(cl.)}$, for $\tilde{l} = 0.001$ and for various values of $\tilde{c}$.

6.2 The imperfection approach

So far, we have concentrated on ideal columns, i.e., the applied force coincides with the centroidal axis of the beam. Next, we shall examine the case of an imperfect column, where the force is applied eccentrically. The main reason for studying the behaviour of columns of imperfect geometries is that by a limiting process, we can determine the behaviour of the perfect system.

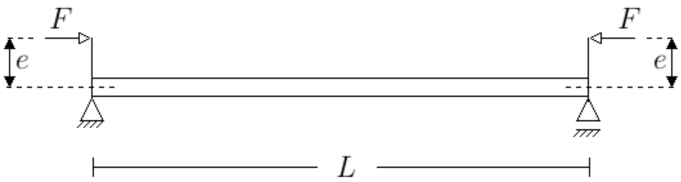


Figure 30: Eccentrically loaded column.

Assume the column of Fig. 30 where the force is applied eccentrically with eccentricity e at both ends. The equilibrium equation in dimensionless form for this case is the same as the one in Eq. (6.28). However, opposite to the boundary conditions (6.27), now the moments $M(0)$, $M(L)$ do not vanish. The boundary conditions in dimensionless form become

$$\tilde{w}(0) = 0\,, \quad \tilde{w}(1) = 0\,, \quad -\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{w}''(0) + \tilde{l}^2\tilde{w}''''(0) = \tilde{k}^2\tilde{e}\,, \tag{6.38}$$

$$-\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{w}''(1) + \tilde{l}^2\tilde{w}''''(1) = \tilde{k}^2\tilde{e}\,, \quad \tilde{w}'''(0) = \tilde{w}'''(1) = 0\,. \tag{6.39}$$

After solving the differential equation, we can establish a load - displacement relation $F_{cr}^{(KG)}/F_{cr}^{(cl.)}$, where $\tilde{\delta}$ is the deflection of the column at mid point $\tilde{x} = 1/2$. This relation is plotted in Fig. 31. We can recognize that as $\tilde{e} \rightarrow 0$, the distributions approach the one of the perfect system and $F_{cr}^{(KG)}$ is the same we have calculated by the eigenvalue method.

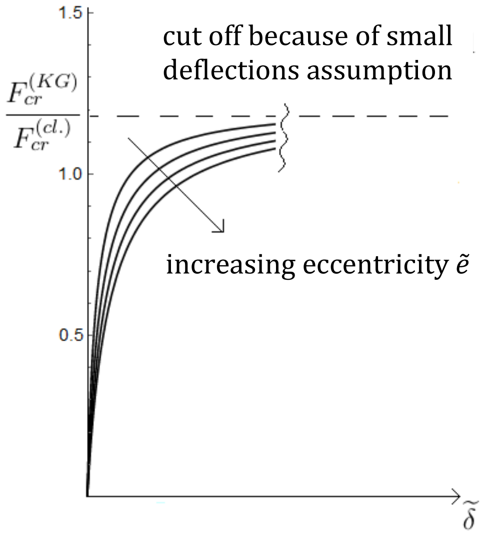


Figure 31: $\left(F_{cr}^{(KG)}/F_{cr}^{(cl.)}\right) - \tilde{\delta}$ diagram with eccentricity effect for a specific value $\tilde{l} = 0.002$.

7 Governing equations and boundary conditions for the KG - Model in dynamics

So far, we have assumed the balance law of linear momentum in the classical form (2.14), i.e., only classical inertial forces $\mathbf{I} = -\rho\ddot{\mathbf{u}}$ were supposed to exist. However, the general setting of gradient elasticity admits the possibility of non - classical inertial forces, besides the classical ones. An elegant method to introduce non - classical inertial forces is to employ the principle of Hamilton. The modifications of Hamilton's principle employed in the thesis have been proposed in [34].

7.1 Gradient elasticity in the setting of Hamilton's principle

It is instructive to recall Hamilton's principle for classical elasticity. For conservative external forces, it reads

$$\delta\int_{t_1}^{t_2}\left[T-\left(\Pi^{(i)}+\Pi^{(cons.ext.)}\right)\right]dt = 0\,, \tag{7.1}$$

where T is the classical kinetic energy,

$$T = T^{(cl.)} := \int_V \frac{1}{2}\rho\dot{\mathrm{u}}_k\dot{\mathrm{u}}_k dV\,, \tag{7.2}$$

and $\Pi^{(i)}$, $\Pi^{(cons.ext.)}$ are the potentials of internal and external forces, respectively. If a part of the external forces is conservative, with potential $\Pi^{(cons.ext.)}$, and the remainder of the external forces expends virtual work $\delta W^{(noncons.ext.)}$, then Eq. (7.1) has to be modified as

$$\delta\int_{t_1}^{t_2}\left[T-\left(\Pi^{(i)}+\Pi^{(cons.ext.)}\right)\right]dt + \int_{t_1}^{t_2}\delta W^{(noncons.ext.)}dt = 0\,. \tag{7.3}$$

As usually, the virtual displacement $\delta\mathbf{u}$ at t_1 and t_2 has to vanish everywhere. Another equivalent formulation of Hamilton's principle (7.3) arises by introducing the classical inertia force

$$\mathrm{i}_k = -\rho\ddot{\mathrm{u}}_k\,, \tag{7.4}$$

and the virtual work $\delta W^{(cl.inert.)}$ of $\mathbf{i}^{(cl.)}$,

$$\delta W^{(cl.inert.)} := \int_V \mathrm{i}_k^{(cl.)}\delta\mathrm{u}_k dV\,, \tag{7.5}$$

and regarding that

$$\delta\int_{t_1}^{t_2} T^{(cl.)}dt = -\int_{t_1}^{t_2}\int_V \rho\ddot{\mathrm{u}}_k\,\delta\mathrm{u}_k dV\,dt = \int_{t_1}^{t_2}\delta W^{(cl.inert.)}dt\,. \tag{7.6}$$

Then, we infer from Eq. (7.3), that

$$-\delta\int_{t_1}^{t_2}\left(\Pi^{(i)}+\Pi^{(cons.ext.)}\right)dt + \int_{t_1}^{t_2}\left(\delta W^{(noncons.ext.)}+\delta W^{(cl.inert.)}\right)dt = 0\,. \tag{7.7}$$

We shall use Hamilton's principle below, in order to derive the equations of motion and the concomitant boundary conditions when non - classical inertial forces are present.

For gradient elasticity, we recall from Eq. (2.11) that the general local balance law of linear momentum, when classical and non - classical body forces **F** are omitted, is of the form

$$\partial_j \Sigma_{jk} = -\mathrm{I}_k \,. \tag{7.8}$$

In Mindlin's gradient elasticity [28], the inertial force **I** is decomposed in classical and non - classical parts,

$$\mathbf{I} = \mathbf{i}^{(cl.)} + \mathbf{i}^{(noncl.)}, \tag{7.9}$$

with $i^{(cl.)}$ as in Eq. (7.4) and

$$\mathrm{i}_k^{(noncl.)} = \frac{1}{3}\partial_p\left(\rho' \tilde{d}^2_{pkmn}\partial_m \ddot{\mathrm{u}}_n\right), \tag{7.10}$$

where ρ' is the mass density of micromaterial per unit macrovolume in the assumed microstructure. By setting

$$\tilde{d}^2_{pkmn} = d^2 \delta_{pm}\delta_{kn} \,, \tag{7.11}$$

$\mathrm{i}_k^{(noncl.)}$ becomes

$$\mathrm{i}_k^{(noncl.)} = \gamma \partial_p \partial_p \ddot{\mathrm{u}}_k = \gamma \Delta \ddot{\mathrm{u}}_k \,, \tag{7.12}$$

where

$$\gamma \coloneqq \frac{1}{3}\rho' d^2 \,. \tag{7.13}$$

In the equations above, d is an internal material length and ρ' has been assumed to be constant. The ansatz (7.11) corresponds to Eq. (9.27) in Mindlin [28], with his α and β^2 being chosen 0 and 1, respectively. From Eqs. (7.8) - (7.13),

$$\partial_j \Sigma_{jk} = \rho \ddot{\mathrm{u}}_k - \gamma \Delta \ddot{\mathrm{u}}_k \,. \tag{7.14}$$

After multiplying Eq. (7.8) by the virtual displacement $\delta \mathrm{u}_k$, and integrating over V,

$$\int_V \partial_j\left(\Sigma_{jk}\delta \mathrm{u}_k\right) dV - \int_V \Sigma_{jk}\delta\epsilon_{jk} dV = -\int_V \mathrm{I}_k \delta \mathrm{u}_k dV \,. \tag{7.15}$$

Next, we replace Σ_{jk} in the second integral with the aid of the constitutive law (3.25),

$$\int_V \partial_j\left(\Sigma_{jk}\delta \mathrm{u}_k\right) dV - \int_V \tau_{jk}\delta\epsilon_{jk} dV + \int_V \left(\partial_m \mu_{mjk}\right)\delta\epsilon_{jk} dV = -\int_V \mathrm{I}_k \delta \mathrm{u}_k dV \,, \tag{7.16}$$

or equivalently,

$$\int_V \partial_j \left(\Sigma_{jk}\delta \mathrm{u}_k + \mu_{jik}(\partial_i \delta \mathrm{u}_k)\right) dV - \int_V \left(\tau_{jk}\delta\epsilon_{jk} + \mu_{mjk}\delta\left(\partial_m \epsilon_{jk}\right)\right) dV = -\int_V \mathrm{I}_k \delta \mathrm{u}_k dV \,, \tag{7.17}$$

and by virtue of Eqs. (3.9) and (3.10),

$$\int_V \partial_j \left(\Sigma_{jk}\delta \mathrm{u}_k + \mu_{jik}(\partial_i \delta \mathrm{u}_k)\right) dV - \delta \int_V \psi dV = -\int_V \mathrm{I}_k \delta \mathrm{u}_k dV \,. \tag{7.18}$$

In order to recast the virtual work expended by the inertial force **I**, appearing as the right - hand side in Eq. (7.18), we substitute from Eqs. (7.9) - (7.13), to obtain

$$-\int_V \mathrm{I}_k \delta \mathrm{u}_k dV = \int_V \rho\, \ddot{\mathrm{u}}_k \delta \mathrm{u}_k dV - \int_V \gamma(\Delta \ddot{\mathrm{u}}_k)\delta \mathrm{u}_k dV$$

$$= \frac{d}{dt}\int_V \rho \dot{\mathrm{u}}_k \delta \mathrm{u}_k dV - \delta \int_V \frac{1}{2}\rho \dot{\mathrm{u}}_k \dot{\mathrm{u}}_k dV - \int_V \gamma(\Delta \ddot{\mathrm{u}}_k)\delta \mathrm{u}_k dV$$

$$= \frac{d}{dt}\int_V \rho \dot{\mathrm{u}}_k \delta \mathrm{u}_k dV - \delta T^{(cl)} - \int_V \gamma(\Delta \ddot{\mathrm{u}}_k)\delta \mathrm{u}_k dV\,, \tag{7.19}$$

where $T^{(cl.)}$ is defined in Eq. (7.2). Recalling that the second integral on the left - hand side of Eq. (7.18) is the virtual work $\delta\Pi^{(i)}$, taking the time integral between t_1 and t_2 of Eq. (7.18) and using the result (7.19),

$$\int_{t_1}^{t_2}\int_V \partial_j \left(\Sigma_{jk}\delta \mathrm{u}_k + \mu_{jik}(\partial_i \delta \mathrm{u}_k)\right) dV\, dt - \delta \int_{t_1}^{t_2} \Pi^{(i)} dt = -\delta \int_{t_1}^{t_2} T^{(cl.)} dt - \int_{t_1}^{t_2}\int_V \gamma(\Delta \ddot{\mathrm{u}}_k)\delta \mathrm{u}_k dV\, dt\,. \tag{7.20}$$

Now, there are two ways to recast the volume integral of the last term on the right - hand side of Eq. (7.20). The one is according to Mindlin and leads to boundary conditions involving inertial terms, whereas the second way, according to Broese et al. [8, 9] leads to boundary conditions with no present inertial terms.

7.2 Gradient elasticity with inertial terms present in the boundary conditions

The aim of Mindlin was to bring Eq. (7.20) to a form corresponding to Eq. (7.1). To this end, the last term on the right - hand side of Eq. (7.20) might be rewritten as follows

$$-\int_{t_1}^{t_2}\int_V \gamma(\Delta \ddot{\mathrm{u}}_k)\mathrm{u}_k dV\, dt = -\int_{t_1}^{t_2}\int_V \gamma\big(\partial_j\, \partial_j \ddot{\mathrm{u}}_k\big)\delta \mathrm{u}_k dV\, dt$$

$$= -\int_{t_1}^{t_2}\int_V \partial_j\big[\gamma\big(\partial_j \ddot{\mathrm{u}}_k\big)\delta \mathrm{u}_k\big] dV\, dt + \int_{t_1}^{t_2}\int_V \gamma\big(\partial_j \ddot{\mathrm{u}}_k\big)\big(\partial_j \delta \mathrm{u}_k\big) dV\, dt$$

$$= -\int_{t_1}^{t_2}\int_V \partial_j\big[\gamma\big(\partial_j \ddot{\mathrm{u}}_k\big)\delta \mathrm{u}_k\big] dV\, dt + \int_{t_1}^{t_2}\frac{d}{dt}\int_V \gamma\big(\partial_p \dot{\mathrm{u}}_k\big)\delta\big(\partial_p \mathrm{u}_k\big) dV\, dt - \int_{t_1}^{t_2}\int_V \gamma\big(\partial_p \dot{\mathrm{u}}_k\big)\delta\big(\partial_p \dot{\mathrm{u}}_k\big) dV\, dt$$

$$= -\int_{t_1}^{t_2}\int_V \partial_j\big[\gamma\big(\partial_j \ddot{\mathrm{u}}_k\big)\delta \mathrm{u}_k\big] dV\, dt - \delta \int_{t_1}^{t_2} T^{(noncl.)} dt, \tag{7.21}$$

where use has been made of the fact that $\partial_p \delta \mathrm{u}_k(t_1) = \partial_p \delta \mathrm{u}_k(t_2) = 0$, and in addition the non - classical kinetic energy

$$T^{(noncl.)} := \int_V \frac{1}{2}\gamma\big(\partial_p \dot{\mathrm{u}}_k\big)\big(\partial_p \dot{\mathrm{u}}_k\big) dV \tag{7.22}$$

has been introduced. From Eqs. (7.20), (7.22),

$$\int_{t_1}^{t_2}\int_V \partial_j\big[\big(\Sigma_{jk} + \gamma \partial_j \ddot{\mathrm{u}}_k\big)\delta \mathrm{u}_k + \mu_{jik}(\partial_i \delta \mathrm{u}_k)\big] dV\, dt - \delta \int_{t_1}^{t_2} \Pi^{(i)} dt$$

$$= -\delta \int_{t_1}^{t_2} T^{(cl.)} dt - \delta \int_{t_1}^{t_2} T^{(noncl.)} dt \, . \tag{7.23}$$

This equation suggests defining total Cauchy stress $\mathbf{\Sigma}^{(T)}$, involving non - classical inertia,

$$\Sigma_{jk}^{(T)} \coloneqq \Sigma_{jk} + \gamma \partial_j \ddot{\mathrm{u}}_k \, , \tag{7.24}$$

and total kinetic energy $T = \overline{T}$,

$$\overline{T} \coloneqq T^{(cl.)} + T^{(noncl.)} \, , \tag{7.25}$$

and to rewrite Eq. (7.23) in the form

$$\int_{t_1}^{t_2} \int_{\partial V} \mathrm{n}_j \left(\Sigma_{jk}^{(T)} \delta \mathrm{u}_k + \mu_{jik} (\partial_i \delta \mathrm{u}_k) \right) dS \, dt - \delta \int_{t_1}^{t_2} \Pi^{(i)} dt = -\delta \int_{t_1}^{t_2} T \, dt \, . \tag{7.26}$$

This in turn suggest to postulate

$$\delta W^{(e)} \equiv \delta \overline{W}^{(e)} \coloneqq \int_{\partial V} \mathrm{n}_j \left(\Sigma_{jk}^{(T)} \delta \mathrm{u}_k + \mu_{jik} (\partial_i \delta \mathrm{u}_k) \right) dS \tag{7.27}$$

as the virtual work of the external forces, and to recast (7.26) as

$$\delta \int_{t_1}^{t_2} \left(\overline{T} - \Pi^{(i)} \right) dt + \int_{t_1}^{t_2} \delta \overline{W}^{(e)} dt = 0 \, , \tag{7.28}$$

which in principle is of the form (7.3)

To accomplish Mindlin's approach, the surface integral $\delta \overline{W}^{(e)}$ must be resolved further, since the gradient $\partial_j \delta \mathrm{u}_k$ is not independent of $\delta \mathrm{u}_k$ on ∂V. After lengthy and elaborate algebraic manipulations, it can be proved (see Mindlin [28] or Broese et al. [8, 9]), that

$$\delta \overline{W}^{(e)} = \int_{\partial \mathrm{V}} \left(\overline{P}_k \delta \mathrm{u}_k + \overline{R}_k (D \delta \mathrm{u}_k) \right) dS \, , \tag{7.29}$$

where, according to Mindlin, $\overline{\mathbf{P}}$ and $\overline{\mathbf{R}}$ are traction vectors defined by

$$\overline{P}_k \coloneqq \mathrm{n}_j \Sigma_{jk}^{(T)} - D\left(\mathrm{n}_i \mu_{ijk}\right) + D_l \mathrm{n}_l \left(\mathrm{n}_i \mathrm{n}_j \mu_{ijk}\right)$$

$$= \mathrm{n}_j \Sigma_{jk} + \gamma \left(\partial_p \ddot{\mathrm{u}}_k\right) \mathrm{n}_p - D\left(\mathrm{n}_i \mu_{ijk}\right) + D_l \mathrm{n}_l \left(\mathrm{n}_i \mathrm{n}_j \mu_{ijk}\right) , \tag{7.30}$$

$$\overline{R}_k \coloneqq \mathrm{n}_i \mathrm{n}_j \mu_{ijk} \, . \tag{7.31}$$

This way, $\delta \overline{W}^{(e)}$ is interpreted as the virtual work expended by the external tractions $\overline{\mathbf{P}}$ and $\overline{\mathbf{R}}$, and Eq. (7.28) is interpreted as the appropriate form of Hamilton's principle for the considered material. Since $\delta \mathbf{u}$ and $D \delta \mathbf{u}$ are independent of each other on ∂V, the adjoint boundary conditions suggested by Eq. (7.29) are

$$\text{either } \overline{P}_k \text{ or } \mathrm{u}_k \text{ and} \tag{7.32}$$

$$\text{either } \overline{R}_k \text{ or } D \mathrm{u}_k \tag{7.33}$$

have to be prescribed on ∂V. For the gradient elasticity theory based on the KG - Model, Eqs. (7.14) are the governing equations of motion, with (7.32), (7.33) being the adjoint boundary conditions proposed by Mindlin. The remarkable feature in the definition of the classical traction in Eq. (7.30) is the presence of inertial terms.

7.3 Gradient elasticity with inertial terms absent in the boundary conditions

In continuum mechanics it is common to require from contact forces to satisfy the principle of material frame indifference. Broese et al. [8, 9] criticized the definition of the traction $\overline{\mathbf{P}}$, as contact forces involving inertial terms cannot satisfy the principle of material frame indifference (see Liu [24, 25]). These authors proposed an alternative but not equivalent definition for the virtual work expended by the external forces. Considering inertial forces to have the nature of conventional body forces (see Gurtin et al. [20], Section 21, and the references cited there), they proposed to recast Eq. (7.20) in the form

$$\delta \int_{t_1}^{t_2} \left(T^{(cl)} - \Pi^{(i)}\right) dt + \int_{t_1}^{t_2} \delta \widehat{W}^{(e)}\, dt + \int_{t_1}^{t_2} \delta \widehat{W}^{(noncl.inert.)}\, dt = 0\,. \tag{7.34}$$

Here, $\delta \widehat{W}^{(e)}$ is defined without inertial terms,

$$\delta \widehat{W}^{(e)} := \int_{\partial V} \left(n_j \Sigma_{jk}^{(P)} \delta \mathrm{u}_k + n_m \mu_{mjk} \left(\partial_j \delta \mathrm{u}_k\right) \right) dS\,, \tag{7.35}$$

with

$$\Sigma_{jk}^{(P)} := \Sigma_{jk} \tag{7.36}$$

being the components of the proper Cauchy stress tensor and $\delta \widehat{W}^{(noncl.inert.)}$ being virtual work expended by the non - classical inertial force $\mathbf{i}^{(noncl.)}$,

$$\delta \widehat{W}^{(noncl.inert.)} := -\int_V \mathrm{i}_k^{(noncl.)} \delta \mathrm{u}_k dV = \int_V \gamma (\Delta \ddot{\mathrm{u}}_k) \delta \mathrm{u}_k dV\,. \tag{7.37}$$

The terms including $T^{(cl.)}$ and $\widehat{W}^{(noncl.inert.)}$ in Eq. (7.34) have to be interpreted to be analogous to corresponding terms due to conservative and non - conservative external forces in Eq. (7.3). By using steps similar to those mentioned in the last section, we can introduce boundary tractions $\widehat{\mathbf{P}}$ and $\widehat{\mathbf{R}}$, so that

$$\delta W^{(e)} \equiv \delta \widehat{W}^{(e)} := \int_{\partial V} \left(\hat{P}_k \delta \mathrm{u}_k + \hat{R}_k \delta (D \mathrm{u}_k) \right) dS\,, \tag{7.38}$$

where (cf. Broese et al. [7])

$$\hat{P}_k := \mathrm{n}_j \Sigma_{jk}^{(P)} - D_j\left(\mathrm{n}_i \mu_{ijk}\right) + D_l \mathrm{n}_l \left(\mathrm{n}_i \mathrm{n}_j \mu_{ijk}\right), \tag{7.39}$$

$$\hat{R}_k := \mathrm{n}_i \mathrm{n}_j \mu_{ijk}\,. \tag{7.40}$$

According to this approach, the proper boundary conditions read

$$\text{either } \hat{P}_k \text{ or } \mathrm{u}_k \text{ and} \tag{7.41}$$

$$\text{either } \hat{R}_k \text{ or } D\mathrm{u}_k \tag{7.42}$$

have to be prescribed on ∂V, where now no inertial terms are present in $\widehat{\mathbf{P}}$. Otherwise, the governing equations of motion are the same as in Eq. (7.14).

8.1 Uniaxial loading of a bar

Let once more the assumptions made in section 4.1.1 for the uniaxial loading of a bar, with length L and cross section A, hold, i.e., $\nu = 0, 2\mu^* = E, \mathrm{x}_1 = \mathrm{x}, \mathrm{u}_1 = \mathrm{u} = \mathrm{u}(\mathrm{x}, t), \mathbf{n} = \mathbf{e}_1, \mathrm{n}_1 = \pm 1, \epsilon_{11} = \epsilon = \epsilon(\mathrm{x}, t) = \mathrm{u}_{,\mathrm{x}}$, so that

$$\Sigma_{11} = \Sigma = E(\mathrm{u}_{,\mathrm{x}} - l^2 \mathrm{u}_{,\mathrm{xxx}}), \tag{8.1}$$

$$\Sigma_{11}^{(T)} = \Sigma^{(T)} = \Sigma + \gamma \mathrm{u}_{,\mathrm{x}tt}, \tag{8.2}$$

$$\Sigma_{11}^{(P)} = \Sigma^{(P)} = \Sigma, \tag{8.3}$$

$$\overline{P}_1 = \overline{P} = \mathrm{n}_1 \Sigma^{(T)} = \mathrm{n}_1(\Sigma + \gamma \mathrm{u}_{,\mathrm{x}tt}), \tag{8.4}$$

$$\hat{P}_1 = \hat{P} = \mathrm{n}_1 \Sigma, \tag{8.5}$$

$$\overline{R}_1 = \overline{R} = \mu_{111} = l^2 \mathrm{u}_{,\mathrm{xx}}. \tag{8.6}$$

Dimensionless forms of equations that we deal with are based on the definitions (cf. Eq. (4.5))

$$\tilde{\mathrm{x}} := \frac{\mathrm{x}}{L}, \qquad \tilde{\mathrm{u}} := \frac{\mathrm{u}}{L}, \qquad \tilde{l} := \frac{l}{L}, \qquad \tilde{t} := \frac{c}{L} t, \qquad \tilde{\gamma} := \frac{\gamma c^2}{E L^2}, \qquad c := \sqrt{\frac{E}{\rho}} \tag{8.7}$$

$$\tilde{\Sigma} := \frac{\Sigma}{E}, \qquad \tilde{\Sigma}^{(T)} := \frac{\Sigma^{(T)}}{E}, \qquad \tilde{\Sigma}^{(P)} := \frac{\Sigma^{(P)}}{E}, \qquad \tilde{\tau} := \frac{P}{EA} \text{ with } P = \tilde{P} \text{ or } \hat{P}. \tag{8.8}$$

The governing equation of motion follows from Eq. (7.14) to be

$$\Sigma_{,\mathrm{x}} = \rho \ddot{\mathrm{u}}_k - \gamma \mathrm{u}_{,\mathrm{xx}tt}, \tag{8.9}$$

or, by virtue of Eqs. (8.1), (8.2),

$$E(\mathrm{u}_{,\mathrm{xx}} - l^2 \mathrm{u}_{,\mathrm{xxxx}}) - \rho \mathrm{u}_{,tt} + \gamma \mathrm{u}_{,\mathrm{xx}tt} = 0. \tag{8.10}$$

The dimensionless form of the latter is

$$\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}} - \tilde{l}^2 \tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}\tilde{\mathrm{x}}\tilde{\mathrm{x}}} - \tilde{\mathrm{u}}_{,\tilde{t}\tilde{t}} + \tilde{\gamma} \tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}\tilde{t}\tilde{t}} = 0 \text{ (KG - Model)}. \tag{8.11}$$

If $\tilde{l} = \tilde{\gamma} = 0$, then

$$\tilde{\mathrm{u}}_{,\tilde{\mathrm{x}}\tilde{\mathrm{x}}} - \tilde{\mathrm{u}}_{,\tilde{t}\tilde{t}} = 0 \text{ (classical elasticity)}, \tag{8.12}$$

which is the governing differential equation in dimensionless form for classical elastic material response. A comprehensive discussion of Eq. (8.11), with respect to size effects and the convergence behaviour for $\tilde{l} \to 0$, has been provided in Broese et al. [7]. Here, the interest is focused on the impact of the classical boundary conditions on the gradient stiffening effect. The non - classical boundary conditions will be the same in all examples, namely vanishing non - classical tractions. The classical boundary condition at $\tilde{\mathrm{x}} = 1$ will be a harmonically with time $\tilde{t}$ varying function of the form $\tilde{A} e^{i\tilde{\omega}\tilde{t}}$, with $\tilde{\omega}$ being a dimensionless operating frequency, $\tilde{A}$ being a displacement - or traction - like amplitude in dimensionless form and i being the imaginary unit. This kind of loading conditions suggests assuming for the solution of Eq. (8.11) to be of the form

$$\tilde{u}(\tilde{x}, \tilde{t}) = \tilde{U}(\tilde{x}) e^{i\tilde{\omega}\tilde{t}}. \tag{8.13}$$

After substituting this in Eq. (8.11) and eliminating the factor $e^{i\tilde{\omega}\tilde{t}}$, we obtain

$$(1 - \tilde{\gamma}\tilde{\omega}^2)\tilde{U}_{,\tilde{x}\tilde{x}} - \tilde{l}^2 \tilde{U}_{,\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \tilde{\omega}^2 \tilde{U} = 0 \quad \text{(KG - Model)}, \tag{8.14}$$

$$\tilde{U}_{,\tilde{x}\tilde{x}} + \tilde{\omega}^2 \tilde{U} = 0 \quad \text{(classical elasticity)}. \tag{8.15}$$

The general solution of the ordinary homogenous differential equation (8.14) is

$$\tilde{U}(\mathrm{x}) = \tilde{C}_1 e^{\xi_1 \tilde{x}} + \tilde{C}_2 e^{-\xi_1 \tilde{x}} + \tilde{C}_3 \cos(\xi_2 \tilde{x}) + \tilde{C}_4 \sin(\xi_2 \tilde{x}), \tag{8.16}$$

with $\tilde{C}_1$,..., $\tilde{C}_4$ being constants of integration and

$$\xi_1 \coloneqq \sqrt{\bar{\xi}_1}, \quad \xi_2 \coloneqq \sqrt{|\bar{\xi}_2|}, \tag{8.17}$$

$$\bar{\xi}_1 \coloneqq \frac{(1 - \tilde{\gamma}\tilde{\omega}^2) + \sqrt{(1 - \tilde{\gamma}\tilde{\omega}^2)^2 + 4\tilde{l}^2\tilde{\omega}}}{2\tilde{l}^2} > 0, \tag{8.18}$$

$$\bar{\xi}_2 \coloneqq \frac{(1 - \tilde{\gamma}\tilde{\omega}^2) - \sqrt{(1 - \tilde{\gamma}\tilde{\omega}^2)^2 + 4\tilde{l}^2\tilde{\omega}}}{2\tilde{l}^2} < 0. \tag{8.19}$$

For the classical case, the corresponding differential equation (8.15) has the general solution

$$\tilde{U} = \tilde{C}_5 \cos \tilde{\omega}\tilde{x} + \tilde{C}_6 \sin \tilde{\omega}\tilde{x}, \tag{8.20}$$

where, once again, $\tilde{C}_5$, $\tilde{C}_6$ are constants of integration. Having available solutions of displacement (8.13), corresponding solutions of strain and stress will be of the form

$$\epsilon = \tilde{U}_{,\tilde{x}}\, e^{i\tilde{\omega}\tilde{t}}, \qquad \tilde{\Sigma} = \tilde{S}(\tilde{x}) e^{i\tilde{\omega}\tilde{t}}, \qquad \tilde{\Sigma}^{(T)} = \tilde{S}^{(T)}(\tilde{x}) e^{i\tilde{\omega}\tilde{t}}, \qquad \tilde{\Sigma}^{(P)} = \tilde{S}^{(P)} e^{i\tilde{\omega}\tilde{t}}, \tag{8.21}$$

with

$$\tilde{S}(\tilde{x}) = \tilde{U}_{,\tilde{x}} - \tilde{l}^2 \tilde{U}_{,\tilde{x}\tilde{x}\tilde{x}}, \tag{8.22}$$

$$\tilde{S}^{(T)}(\tilde{x}) = (1 - \tilde{\gamma}\tilde{\omega}^2)\tilde{U}_{,\tilde{x}} - \tilde{l}^2 \tilde{U}_{,\tilde{x}\tilde{x}\tilde{x}}, \tag{8.23}$$

$$\tilde{S}^{(P)}(\tilde{x}) = \tilde{S}^{(P)} = \tilde{S} = \tilde{U}_{,\tilde{x}} - \tilde{l}^2 \tilde{U}_{,\tilde{x}\tilde{x}\tilde{x}}. \tag{8.24}$$

In the following, we will present two different cases. In the first case, non - classical inertial terms are vanishing and in the second one they are present. In addition, for the second case, we shall examine a version where non - classical inertial terms are present in both the differential equation and the boundary conditions and a version where these terms are present only in the differential equation but not in the boundary conditions. For every example, calculations have been made for both traction and displacement controlled classical loading conditions, at $\tilde{x} = 1$.

8.2 Responses predicted by the KG - Model without non - classical inertial terms

This case has also been discussed in Broese et al. [8, 9] but with non -classical boundary conditions different from those in the present thesis. First, we consider the case of absent non - classical inertial terms, i.e., $\tilde{\gamma} = 0$. The bar is assumed to be fixed at its left end, $\tilde{x} = 0$, and subject to harmonically varying, with time $\tilde{t}$, classical traction at its right end (traction - controlled boundary condition)

$$\tilde{\tau} = \tilde{\tau}_0 e^{i\tilde{\omega}\tilde{t}} , \tag{8.25}$$

with $\tilde{\tau}_0 = \text{const}$. At both ends, the non - classical tractions are vanishing. Keeping in mind Eqs. (8.1) - (8.8), the imposed boundary conditions might be expressed as follows:

$$[\tilde{u}]_{(\tilde{x}=0,\tilde{t})} = 0 , \qquad [\tilde{u}_{,\tilde{x}} - \tilde{l}^2 \tilde{u}_{,\tilde{x}\tilde{x}\tilde{x}}]_{(\tilde{x}=1,\tilde{t})} = \tilde{\tau}_0 e^{i\tilde{\omega}\tilde{t}} , \tag{8.26}$$

$$[\tilde{u}_{,\tilde{x}\tilde{x}}]_{(\tilde{x}=0,\tilde{t})} = 0 , \qquad [\tilde{u}_{,\tilde{x}\tilde{x}}]_{(\tilde{x}=1,\tilde{t})} = 0 , \tag{8.27}$$

and by elaborating Eq. (8.13) and eliminating the factor $e^{i\tilde{\omega}\tilde{t}}$,

$$[\tilde{U}]_{\tilde{x}=0} = 0 , \qquad [\tilde{U}_{,\tilde{x}} - \tilde{l}^2 \tilde{U}_{,\tilde{x}\tilde{x}\tilde{x}}]_{\tilde{x}=1} = \tilde{\tau}_0 , \tag{8.28}$$

$$[\tilde{U}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=0} = 0 , \qquad [\tilde{U}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=1} = 0 . \tag{8.29}$$

The corresponding boundary conditions for the classical case are

$$[\tilde{U}]_{\tilde{x}=0} = 0 , \qquad [\tilde{U}_{,\tilde{x}}]_{\tilde{x}=1} = \tilde{\tau}_0 . \tag{8.30}$$

Solutions of (8.14) for the KG - Model, and of (8.15) for the classical elasticity, are derived from Eq. (8.16) - (8.20), with the integration constants adapted to the above boundary conditions. Resulting distributions of $\tilde{U}$, $\tilde{U}_{,\tilde{x}}$, $\tilde{S}$ are illustrated in Figs. 32, 33, for $\tilde{\tau}_0 = 0.0005$ and two different frequencies $\tilde{\omega}$.

For sufficiently small values of $\tilde{\omega}$ (see, e.g., Fig. 32 a)), all $\tilde{U}$ – distributions predicted by the KG – Model are monotonically increasing, they do not intersect for $\tilde{x} > 0$ and indicate the gradient stiffening effect in comparisson to the classical solution. In particular, the stiffening effect is increasing with increasing values of $\tilde{l}$. The corresponding $\tilde{S}$ – distributions, shown in Fig. 32 c), are monotonically decreasing and they do not intersect for $\tilde{x} < 1$. It is worth mentioning that all $\tilde{S}$ – distributions are below the classical one and that the strain distributions $\tilde{U}_{,\tilde{x}}$ intersect at a point $\tilde{x} \in (0,1)$. Thus, the distributions of $\tilde{U}$, $\tilde{U}_{,\tilde{x}}$ and $\tilde{S}$ indicate a common intersection point, respectively.

However, from Fig. 33 and further results not reported here, we can conclude that no regular tendencies, or even the opposite, may happen for sufficiently large values of $\tilde{\omega}$. Another qualitative difference is that new intersection points in the distributions can occur, and positions of intersection points can change depending on the applied frequency. In particular, for sufficiently large frequencies, it may happen that $\tilde{U}_{,\tilde{x}}$ distributions do not exhibit common intersection points at all.

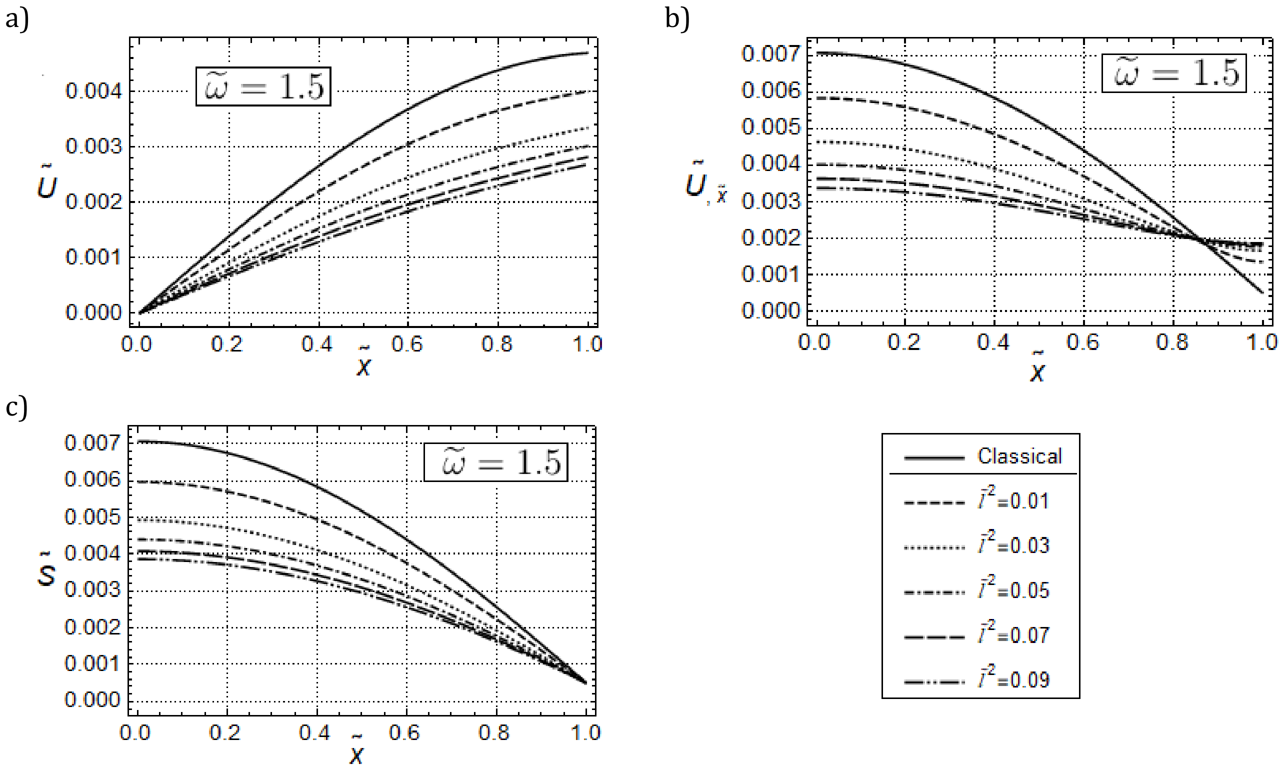


Figure 32: Isotropic KG-Model, vanishing non - classical inertial terms. Predicted distributions of amplitudes a) $\tilde{U}$, b) $\tilde{U}_{,\tilde{x}}$ and c) $\tilde{S} \equiv \tilde{S}^{(T)} \equiv \tilde{S}^{(P)}$ for various values of $\tilde{l}$ with $\widetilde{\omega} = 1.5$.

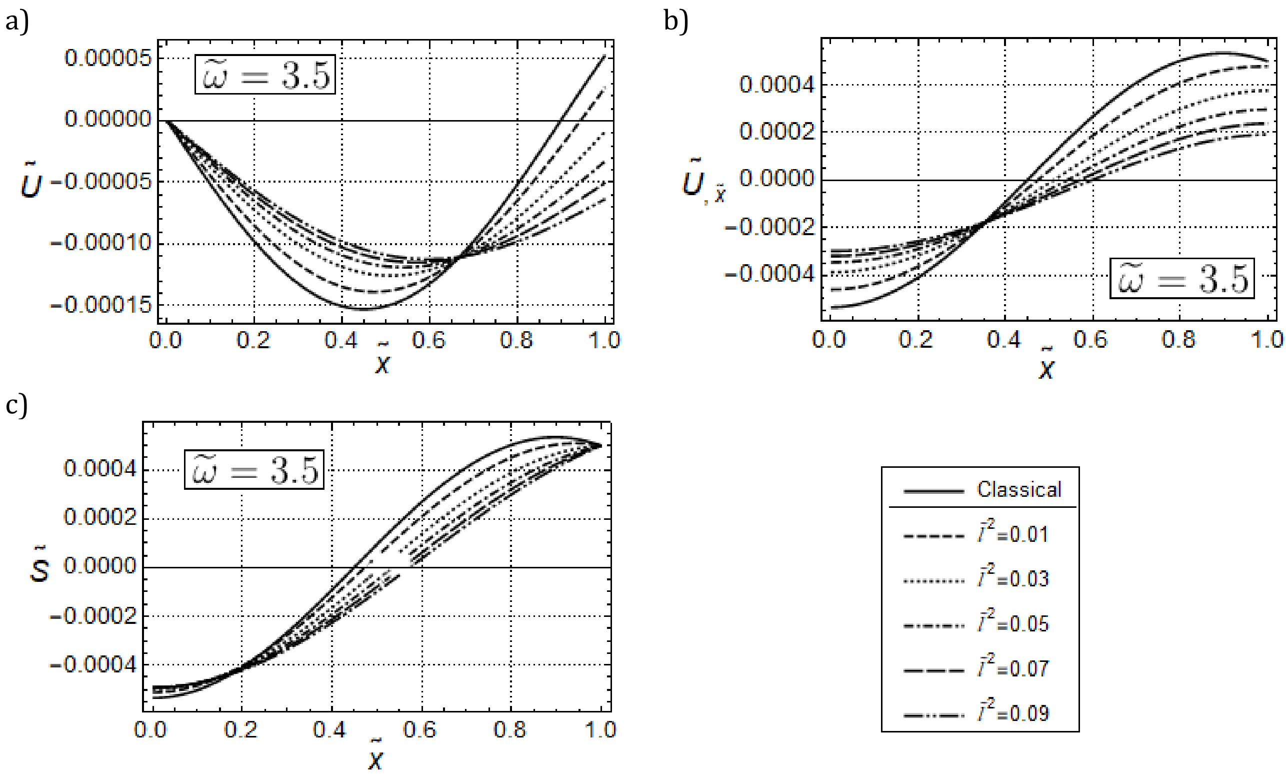


Figure 33: Isotropic KG-Model, vanishing non - classical inertial terms. Predicted distributions of amplitudes a) $\tilde{U}$, b) $\tilde{U}_{,\tilde{x}}$ and c) $\tilde{S} \equiv \tilde{S}^{(T)} \equiv \tilde{S}^{(P)}$ for various values of $\tilde{l}$ with $\widetilde{\omega} = 3.5$.

Next, we consider the case of displacement - controlled boundary conditions, which arises from (8.26), (8.27) by replacing $(8.26)_2$ with an appropriate prescription, i.e.,

$$[\tilde{u}]_{(\tilde{x}=0,\tilde{t})} = 0\,, \qquad [\tilde{u}]_{(\tilde{x}=1,\tilde{t})} = \tilde{u}_0 e^{i\tilde{\omega}\tilde{t}}\,, \tag{8.31}$$

$$[\tilde{u}_{,\tilde{x}\tilde{x}}]_{(\tilde{x}=0,\tilde{t})} = 0\,, \qquad [\tilde{u}_{,\tilde{x}\tilde{x}}]_{(\tilde{x}=1,\tilde{t})} = 0\,, \tag{8.32}$$

with $\tilde{u}_0$ being a constant displacement amplitude. After eliminating the factor $e^{i\tilde{\omega}\tilde{t}}$, these boundary conditions reduce to

$$[\tilde{U}]_{\tilde{x}=0} = 0\,, \qquad [\tilde{U}]_{\tilde{x}=1} = \tilde{u}_0\,, \tag{8.33}$$

$$[\tilde{U}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=0} = 0\,, \qquad [\tilde{U}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=1} = 0\,. \tag{8.34}$$

Predicted distributions for the KG – Model and classical elasticity for the above displacement – controlled boundary conditions are depicted in Figs. 34, 35 for frequencies $\tilde{\omega} = 1.5$ and $\tilde{\omega} = 3.5$ and $\tilde{u}_0 = 0.005$. The general observations concerning $\tilde{U}$ responses are similar to those for traction – controlled loading. Clearly, due to the imposed boundary conditions, the $\tilde{U}$ – distributions intersect now at $\tilde{x} = 1$ as well. It can be recognized from Fig. 34, that for sufficiently small values of $\tilde{\omega}$ only small quantitative differences are visible, which could be expected because of the assumed dislpacement boundary conditions. In the related $\tilde{S}$ – distributions for small values of $\tilde{\omega}$ only small quantitative differences are visible as well. However, there is the remarkable qualitative difference that now, these distributions do intersect for some $0 < \tilde{x} < 1$.

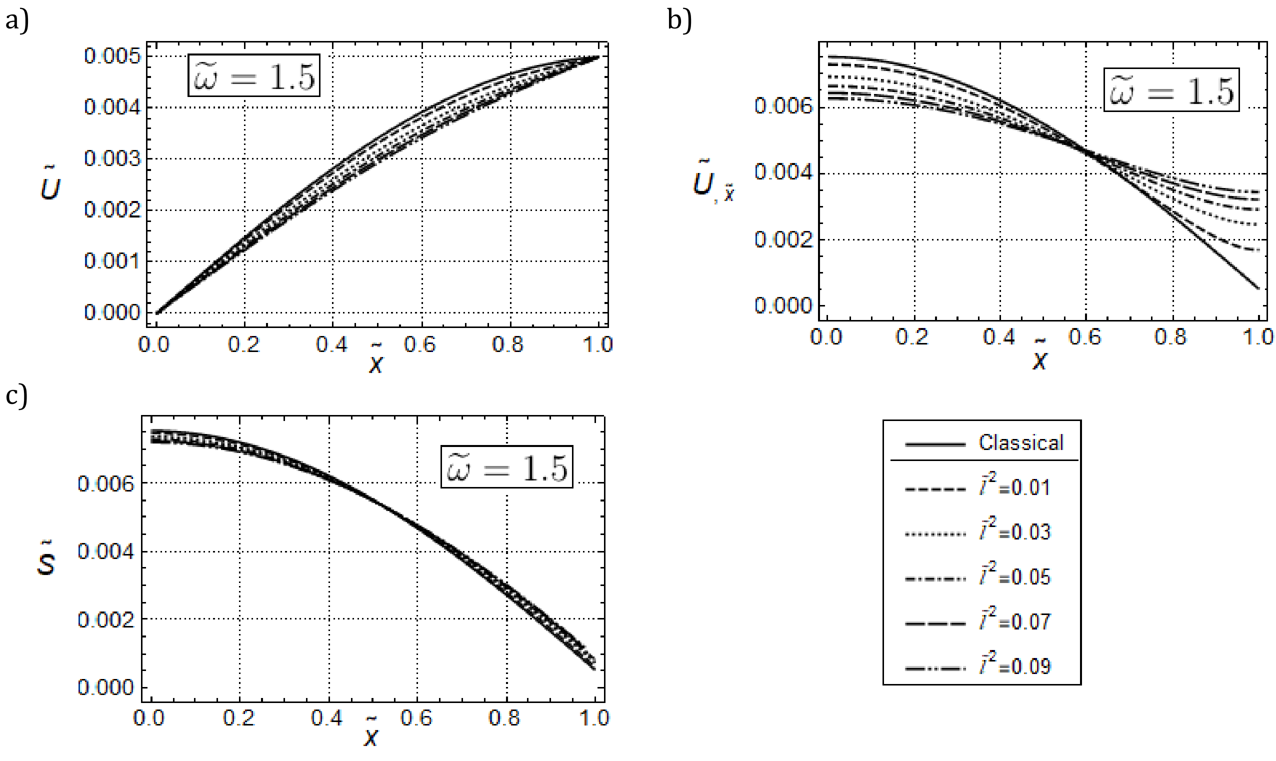


Figure 34: Isotropic KG-Model, vanishing non - classical inertial terms. Predicted distributions of amplitudes a) $\tilde{U}$, b) $\tilde{U}_{,\tilde{x}}$ and c) $\tilde{S} \equiv \tilde{S}^{(T)} \equiv \tilde{S}^{(P)}$ for various values of $\tilde{l}$ with $\tilde{\omega} = 1.5$.

For sufficiently large values of $\tilde{\omega}$ (see Fig. 35), no regular tendencies in the predicted responses can be detected.

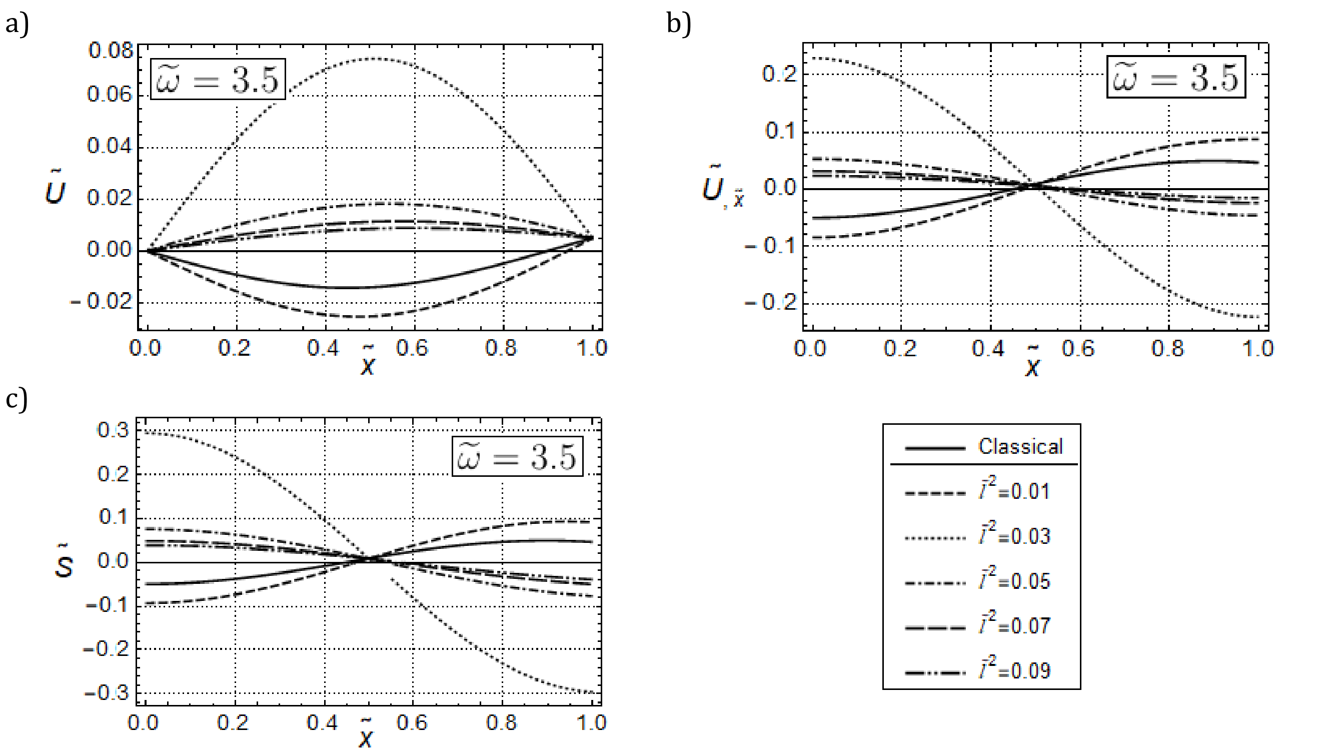


Figure 35: Isotropic KG-Model, vanishing non - classical inertial terms. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S} \equiv \tilde{S}^{(T)} \equiv \tilde{S}^{(P)}$ for various values of $\tilde{l}$ with $\widetilde{\omega} = 3.5$.

8.3 Responses predicted by the KG - Model when non - classical inertial terms are present

Here, non - classical inertial terms are present in the governing equation of motion, i.e., differential equation (8.14) has to be solved for $\tilde{\gamma} \neq 0$. In addition, two differrent versions will be examined, depending on whether inertial terms are present in the boundary conditions or not.

Version 1

Assume non - classical inertial terms to be present in the imposed boundary conditions (cf. Eqs (7.30) - (7.33)). Then the boundary conditions corresponding to (8.26), (8.27) are

$$[\tilde{u}]_{(\tilde{x}=0,\tilde{t})} = 0\,, \qquad [\tilde{u}_{,\tilde{x}} - \tilde{l}^2 \tilde{u}_{,\tilde{x}\tilde{x}\tilde{x}} + \tilde{\gamma}\tilde{u}_{,\tilde{x}\tilde{t}\tilde{t}}]_{(\tilde{x}=1,\tilde{t})} = \tilde{\tau}_0 e^{i\widetilde{\omega}\tilde{t}}\,, \tag{8.35}$$

$$[\tilde{u}_{,\tilde{x}\tilde{x}}]_{(\tilde{x}=0,\tilde{t})} = 0\,, \qquad [\tilde{u}_{,\tilde{x}\tilde{x}}]_{(\tilde{x}=1,\tilde{t})} = 0\,. \tag{8.36}$$

After eliminating the factor $e^{i\widetilde{\omega}\tilde{t}}$, by utilizing Eq. (8.13)

$$[\widetilde{U}]_{\tilde{x}=0} = 0\,, \qquad [(1-\tilde{\gamma}\widetilde{\omega}^2)\widetilde{U}_{,\tilde{x}} - \tilde{l}^2 \widetilde{U}_{,\tilde{x}\tilde{x}\tilde{x}}]_{\tilde{x}=1} = \tilde{\tau}_0\,, \tag{8.37}$$

$$[\widetilde{U}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=0} = 0\,, \qquad [\widetilde{U}_{,\tilde{x}\tilde{x}}]_{\tilde{x}=1} = 0\,, \tag{8.38}$$

where $\tilde{\tau}_0 =$ const. is the amplitude of a classical traction applied at the right end of the bar (traction – controlled boundary condition). On solving Eq. (8.14) with the aid of the boundary conditions (8.37), (8.38), we obtain distributions of $\widetilde{U}$, $\widetilde{U}_{,\tilde{x}}$, $\tilde{S}^{(T)}$, for various values of $\tilde{\gamma}$ and for $\tilde{\tau}_0 = 0.0005$, $\tilde{l} = 0.2$ and two different frequencies $\widetilde{\omega}$, as illustrated in Figs. 36, 37.

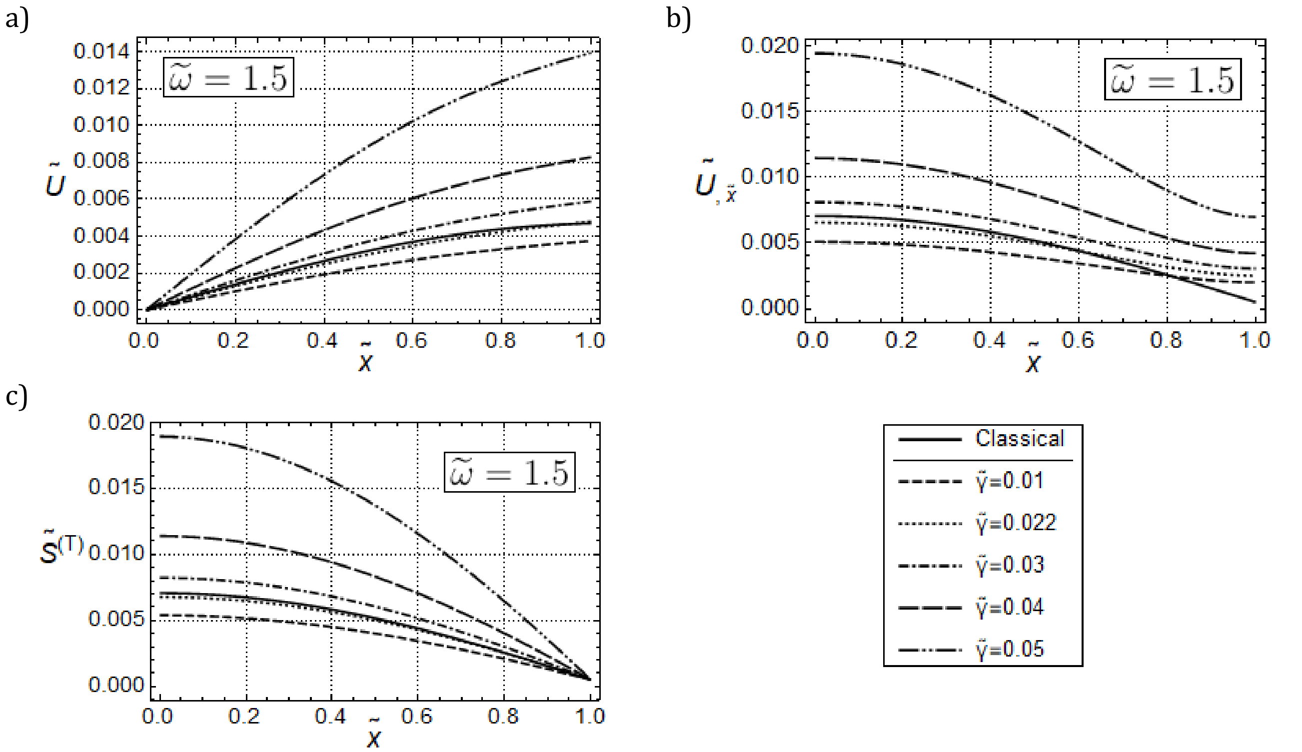


Figure 36: Isotropic KG-Model, Version 1 for traction – controlled loading conditions. Predicted distributions of amplitudes a) $\tilde{U}$, b) $\tilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(T)}$ for various values of $\tilde{\gamma}$ with $\tilde{l} = 0.2$, $\widetilde{\omega} = 1.5$.

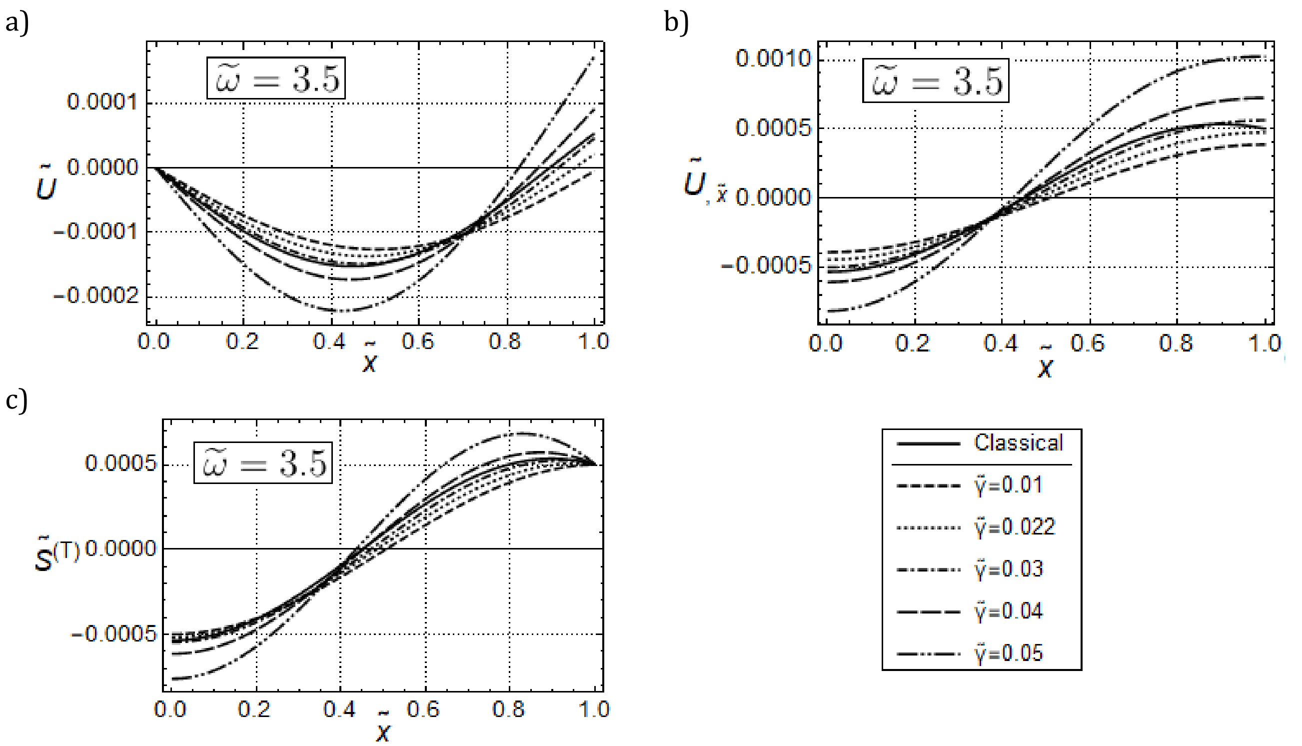


Figure 37: Isotropic KG-Model, Version 1 for traction – controlled loading conditions. Predicted distributions of amplitudes a) $\tilde{U}$, b) $\tilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(T)}$ for various values of $\tilde{\gamma}$ with $\tilde{l} = 0.2$, $\widetilde{\omega} = 3.5$.

It is apparent (see Fig. 36), that for sufficiently small frequencies $\tilde{\omega}$, by increasing values of $\tilde{\gamma}$ the $\widetilde{U}$ – distributions predicted by the KG - Model are also increasing and can exceed the one predicted by classical elasticity, which in turn indicates that both gradient stiffening and softening effect can occur, depending on the values of $\tilde{\gamma}$. This behaviour carries over to the $\tilde{S}^{(T)}$ distributions as well. The amount of stresses predicted by the KG – Model is also increasing with increasing values of $\tilde{\gamma}$ (see Fig. 36 c)). Another qualitative difference between Version 1 and the results in section 8.2 (cf. Fig. 32) is that, now, additional intersection points for $\widetilde{U}$ – and $\tilde{S}^{(T)}$– distributions exist for $\tilde{x} \in (0,1)$.

Next, we consider the case of displacement – controlled boundary conditions, which are identical to boundary conditions (8.31) - (8.34). Predicted distributions for the KG - Model and classical elasticity for the displacement – controlled boundary conditions are depicted in Figs. 38, 39, for $\tilde{u}_0 = 0.005$, $\tilde{l} = 0.2$ and for frequencies $\tilde{\omega} = 1.5$ and $\tilde{\omega} = 3.5$. For sufficiently small frequencies $\tilde{\omega}$, concerning $\widetilde{U}$ responses, only small quantitative differences are visible. After magnification of Fig. 38 a), it was made clear that these distributions increase with increasing values of $\tilde{\gamma}$ but are always below the one predicted by classical elasticity. On the other hand, observing Fig. 38 c) closely, shows that $\tilde{S}^{(T)}$ – distributions predicted by the KG – Model predicts decrease with increasing values of $\tilde{\gamma}$ and these distributions intersect the one predicted by classical elasticity for $\tilde{x} \in (0,1)$.

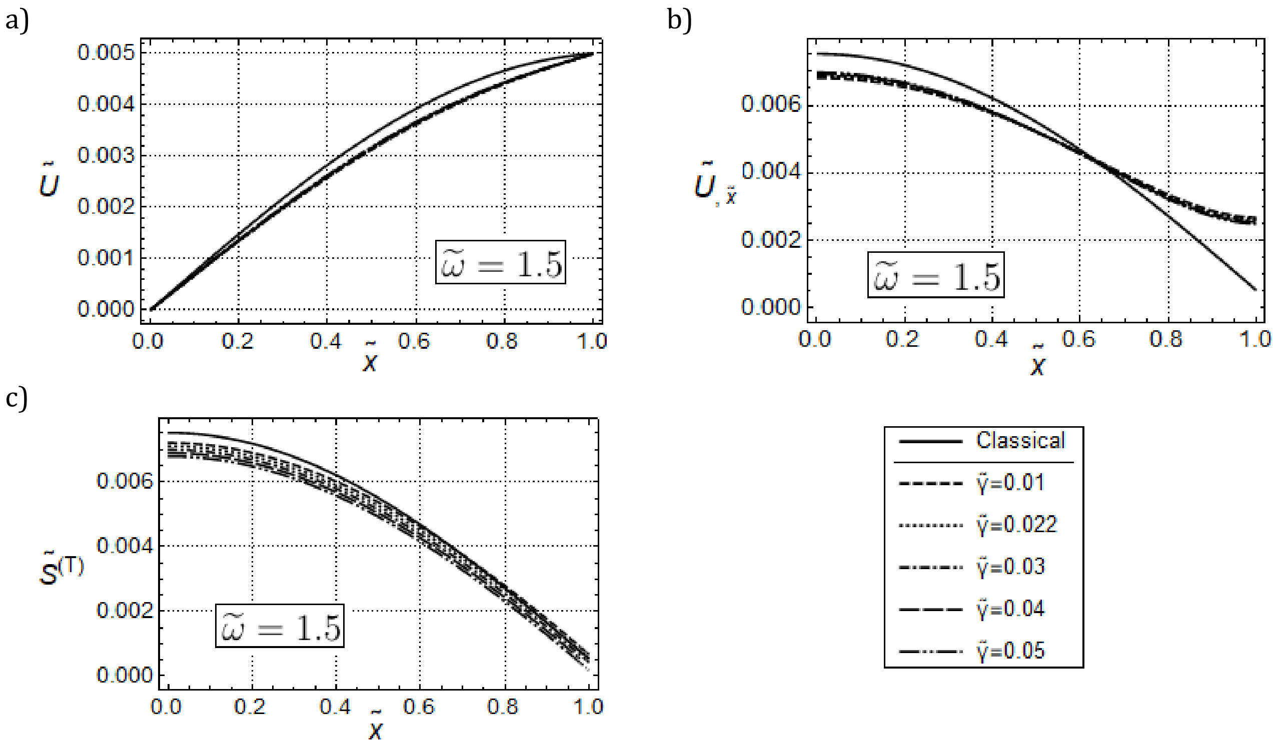


Figure 38: Isotropic KG-Model, Version 1 for displacement – controlled loading conditions. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(T)}$ for various values of $\tilde{\gamma}$ with $\tilde{l} = 0.2$, $\tilde{\omega} = 1.5$.

For higher frequencies, the responses (see Figs.37, 39) the responses seem somewhat similar to those in Section 8.2. However, the arrangement of the responses with respect to the classical one differs.

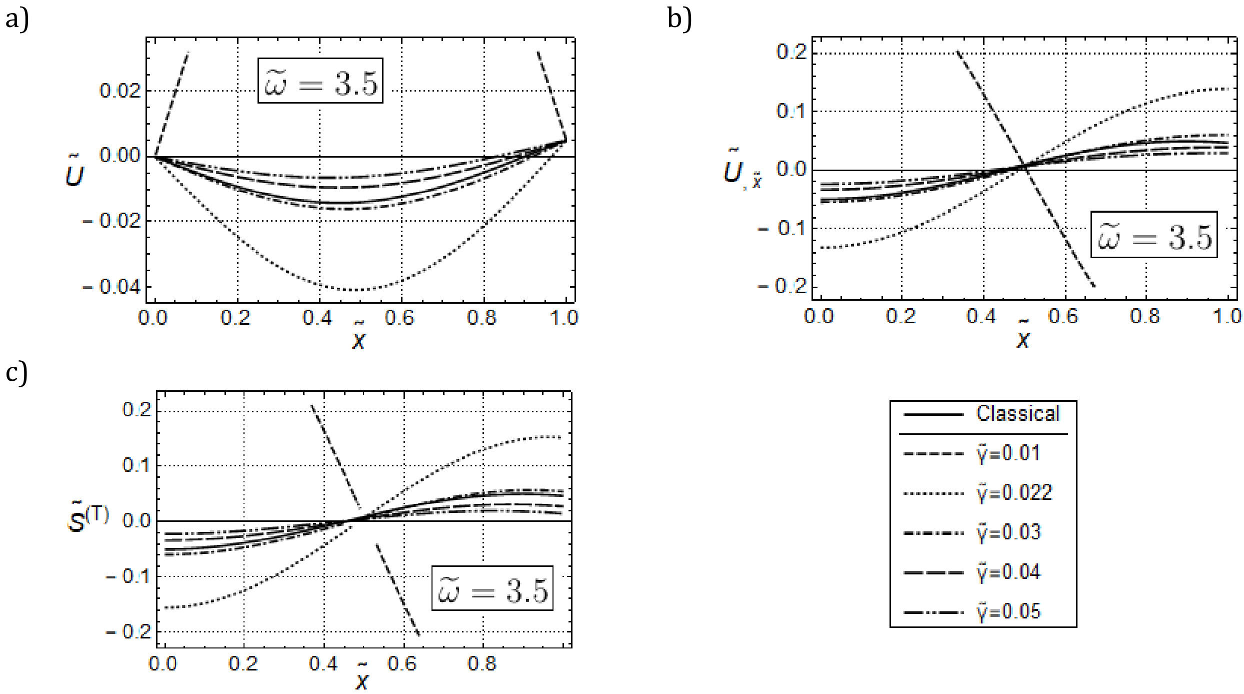


Figure 39: Isotropic KG-Model, Version 1 for displacement – controlled loading conditions. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(T)}$ for various values of $\tilde{\gamma}$ with $\tilde{l} = 0.2$, $\widetilde{\omega} = 3.5$.

Version 2

Assume now non - classical inertial terms to be absent in the imposed boundary conditions (cf. Eqs. (7.39) – (7.42)). Then the boundary conditions for traction – controlled loading in Eqs. (8.26) - (8.29) remain the same. Distributions of $\widetilde{U}$, $\widetilde{U}_{,\tilde{x}}$ and $\tilde{S}^{(P)}$ for various values of $\tilde{\gamma}$ and for $\tilde{\tau}_0 = 0.0005$, $\tilde{l} = 0.2$ and two different frequencies $\widetilde{\omega} = 1.5$ and $\widetilde{\omega} = 3.5$ are displayed in Figs. 40, 41.

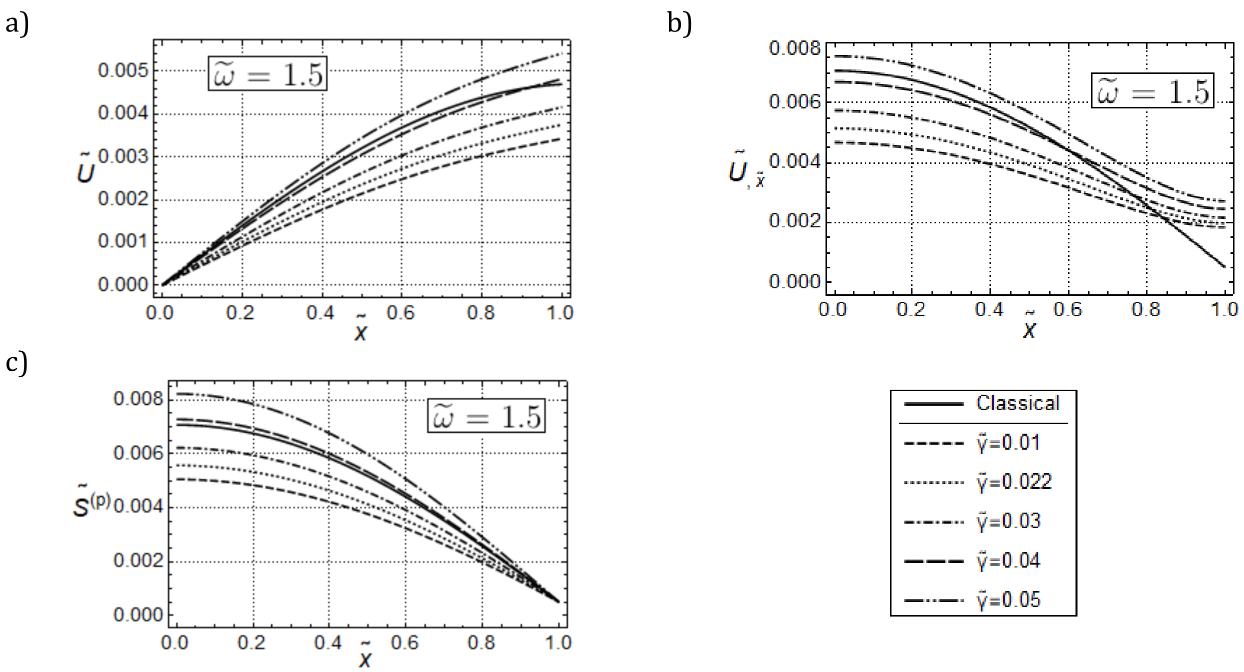


Figure 40: Isotropic KG-Model, absent non - classical inertial terms in boundary conditions. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(P)}$ for various values of $\tilde{\gamma}$ with $\widetilde{\omega} = 1.5$.

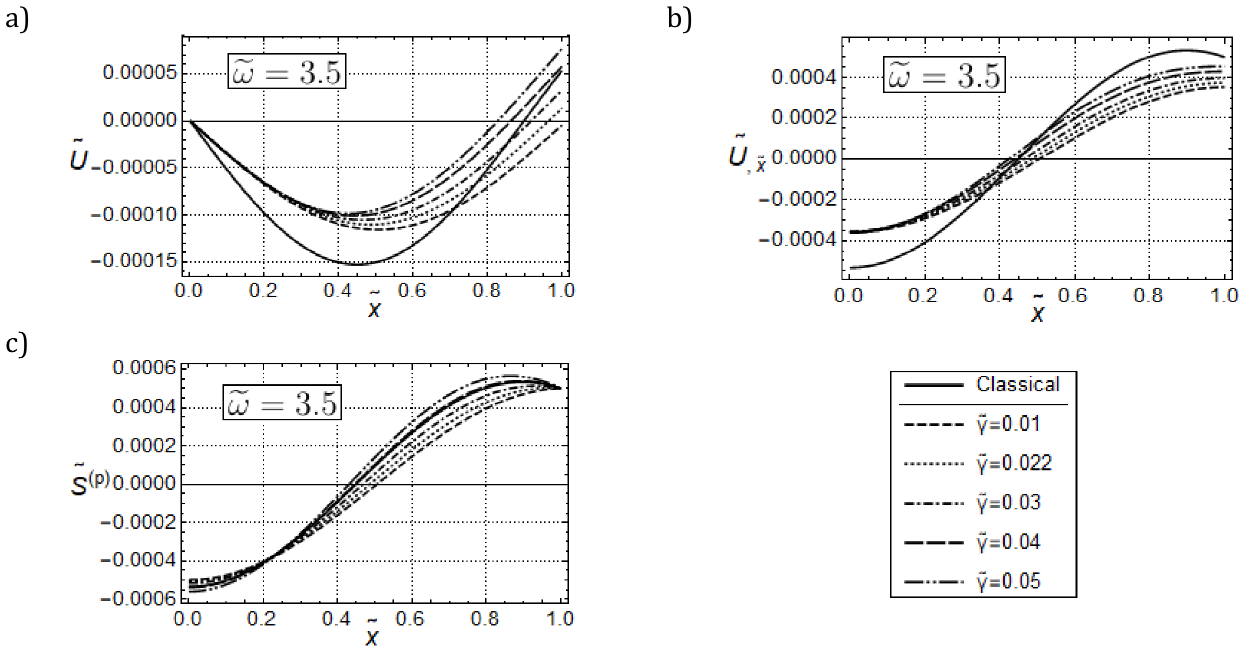


Figure 41: Isotropic KG-Model, absent non - classical inertial terms in boundary conditions. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(P)}$ for various values of $\tilde{\gamma}$ with $\widetilde{\omega} = 3.5$.

As seen in Fig. 40, for sufficiently small frequencies and keeping $\tilde{l}$ constant, all the $\widetilde{U}$, $\widetilde{U}_{,\tilde{x}}$ and $\tilde{S}^{(P)}$ distributions predicted by the KG - Model are increasing with increasing values of $\tilde{\gamma}$. These observations are similar to the ones made for Version 1 (cf. Fig. 36). However, the amount of these distributions is smaller for Version 2, i.e., for a certain amount of $\tilde{\gamma}$, Version 2 predicts smaller displacement and strain distributions compared to Version 1 and $\tilde{S}^{(P)} < \tilde{S}^{(T)}$ as well (see Figs. 42 - 44). It is worth mentioning that for displacement – controlled boundary conditions, both Versions 1 and 2 predict the same $\widetilde{U}$, $\widetilde{U}_{,\tilde{x}}$ and differ only in the distributions of stresses. Comparisons of the predicted responses in the context of Versions 1, 2 are illustrated in Figs. 42 - 44. From Fig. 44 a) it is clear, that for $\widetilde{\omega} = 1.5$, Version 1 predicts smaller stresses than Version 2 and in addition, by increasing the value of $\tilde{\gamma}$, Version 1 distributions decrease, whereas Version 2 distributions are increasing. This observation is the opposite of the one made in Fig. 42 c). Concluding this section, these results do not allow to reject one version over the other based on plausibility arguments concerning the predicted responses.

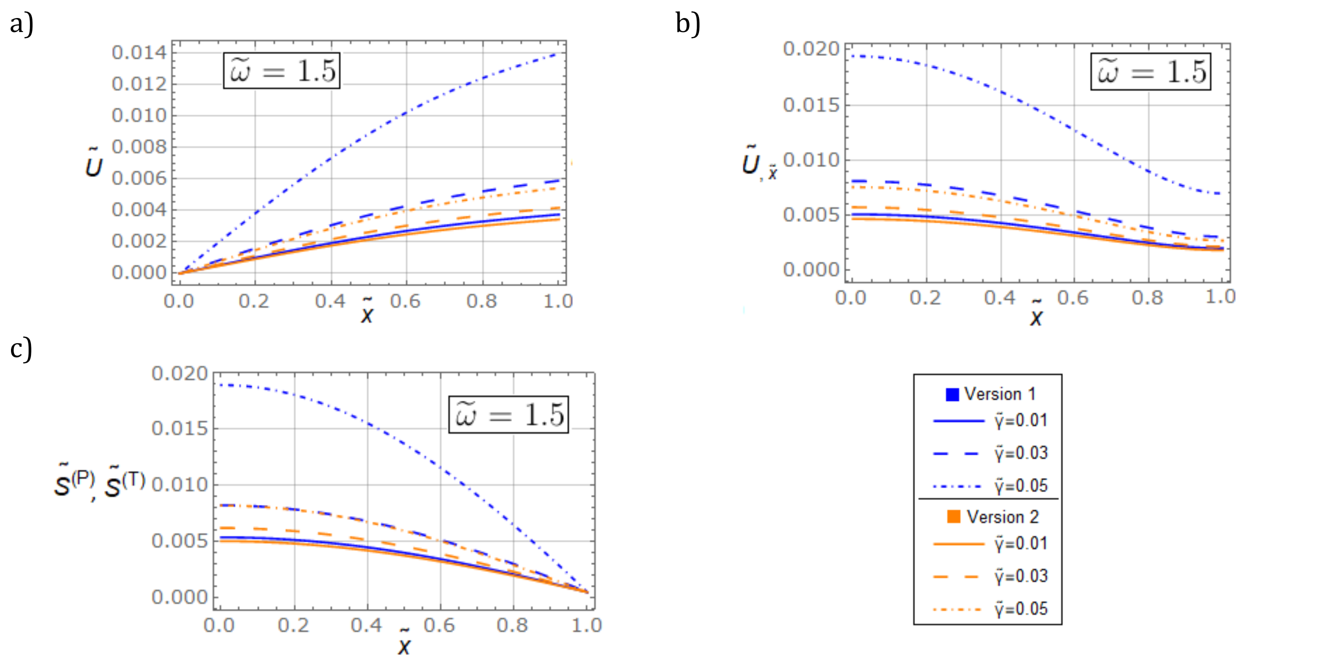


Figure 42: Comparison of Versions 1 and 2, for traction – controlled boundary conditions. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(T)}$ and $\tilde{S}^{(P)}$ for various values of $\tilde{\gamma}$ with $\widetilde{\omega} = 1.5$.

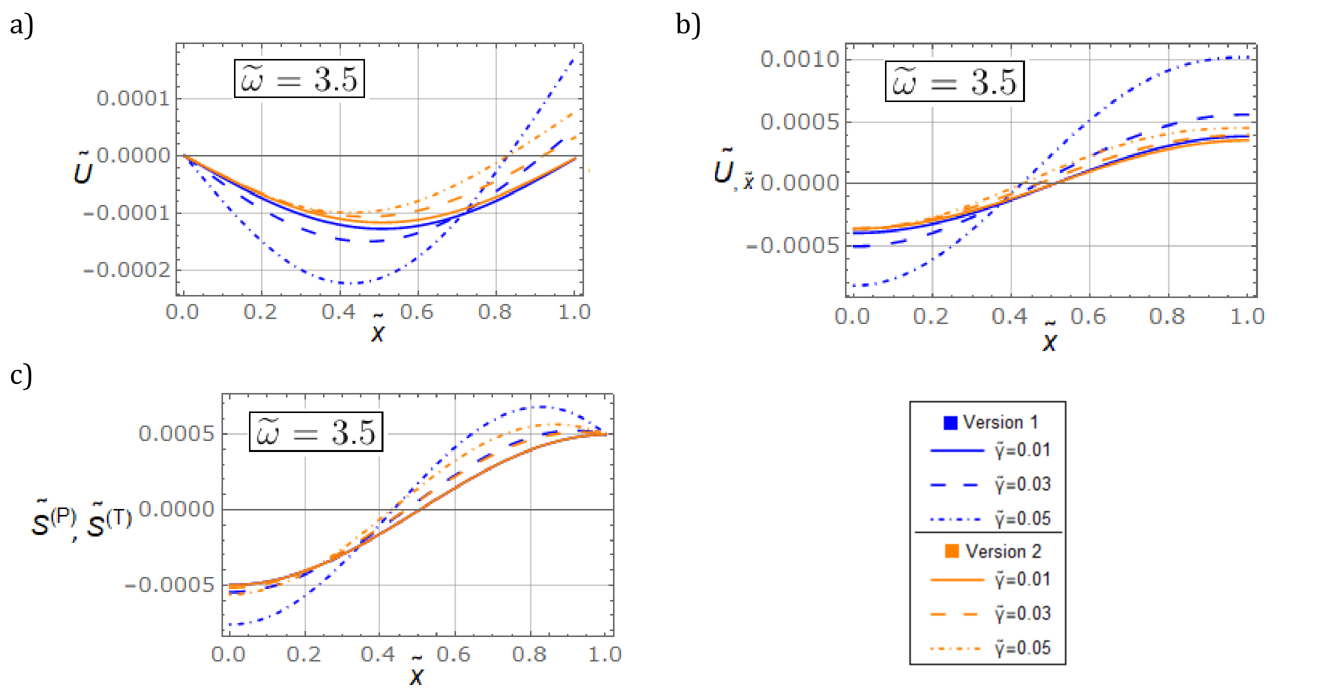
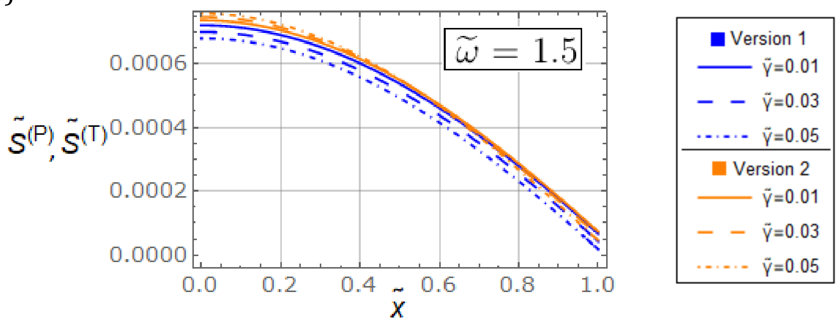


Figure 43: Comparison of Versions 1 and 2, for traction – controlled boundary conditions. Predicted distributions of amplitudes a) $\widetilde{U}$, b) $\widetilde{U}_{,\tilde{x}}$ and c) $\tilde{S}^{(T)}$ and $\tilde{S}^{(P)}$ for various values of $\tilde{\gamma}$ with $\widetilde{\omega} = 3.5$.

a)



b)

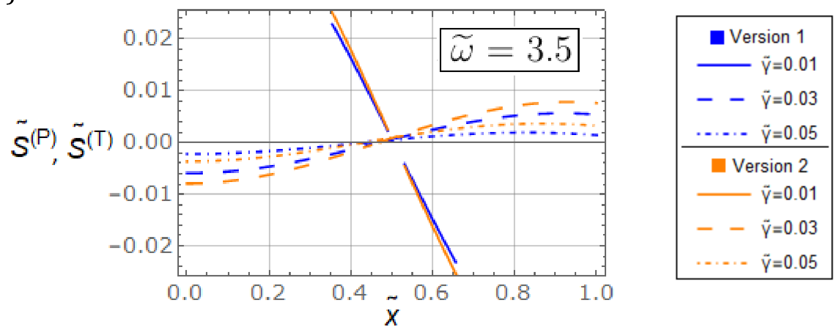


Figure 44: Comparison of Versions 1 and 2, for displacement – controlled boundary conditions. Predicted distributions of amplitudes $\tilde{S}^{(T)}$ and $\tilde{S}^{(P)}$ for various values of $\tilde{\gamma}$ with a) $\widetilde{\omega} = 1.5$ and b) $\widetilde{\omega} = 3.5$.

9 Euler - Bernoulli beam in dynamics

The aim of this section is to generalize for dynamical problems the developed static Euler - Bernoulli beam theory in the framework of the KG - Model. For definiteness, and for simplicity, we shall do this for a cantilever beam undergoing transverse vibrations. For our purposes it suffices to represent the theory in terms of section forces only, i.e., the relation to stress components will not be addressed.

It is clear that the virtual work expended by external forces for a three - dimensional body exhibiting material properties specified by the KG - Model is expressible in terms of classical traction **P** and non - classical traction **R**. This holds true for both statical and dynamical problems. Further, it has been clear in the last section, that if non - classical inertial forces are assumed to exist, then two different versions of Hamilton's principle may be adopted. The first version arises whenever the kinetic energy of the body is defined to be composed of a classical and a non - classical part. In this case, the principle of Hamilton enforces the classical boundary traction to include inertial terms, i.e., $P = \overline{P}$. The second version supposes the kinetic energy of the body to be the classical one. Hamilton's principle for this case does not require the presence of inertial terms in the classical boundary traction, i.e., $P = \hat{P}$. In this section we shall show how these issues are reflected to the Euler - Bernoulli beam theory. More specifically, we shall show that the sectional constitutive law for moment may or may not depend on inertial terms. Most parts of the theory in this section has been proposed in [34].

9.1 Hamilton's principles for Euler - Bernoulli beams

Again, let all assumptions made in section 5.1.2 apply and in addition suppose for the coordinate system, that the neutral axis coincides with the centroidal one. This is the semi - inverse approach of classical elasticity, where some part of the solution is supposed and remain valid as long as there are no contradictions. Independent on whether non - classical inertial terms are present or not, the virtual work of the external forces, $\delta W^{(e)}$, and of the internal forces $\delta\Pi^{(i)}$, according to Eqs. (5.255) and (5.248), (5.249) are

$$\delta W^{(e)} = \int_{\partial \mathrm{V}} \big(\mathbf{P}\cdot\delta\mathrm{u} + \mathbf{R}\cdot\delta(D\mathbf{u})\big)dS = [Q\delta\mathrm{w}]_{\mathrm{x}=0}^{\mathrm{x}=L} + [M(-\delta\mathrm{w}')]_{\mathrm{x}=0}^{\mathrm{x}=L} + [m(-\delta\mathrm{w}'')]_{\mathrm{x}=0}^{\mathrm{x}=L} + \int_0^L q\,\delta\mathrm{w}\,d\mathrm{x}\,, \quad (9.1)$$

$$\delta\Pi^{(i)} = [f_1(-\delta\mathrm{w}')]_{\mathrm{x}=0}^{\mathrm{x}=L} + [f_1'\delta\mathrm{w}]_{\mathrm{x}=0}^{\mathrm{x}=L} + [f_2(-\delta\mathrm{w}'')]_{\mathrm{x}=0}^{\mathrm{x}=L} + \int_0^L f_3\delta\mathrm{w}\,d\mathrm{x}\,, \quad (9.2)$$

with

$$f_1 = -[(EI + l^2EA)\mathrm{w}'' - l^2EI\mathrm{w}'''']\,, \quad (9.3)$$

$$f_2 = -l^2EI\mathrm{w}'''\,, \quad (9.4)$$

$$f_3 = (EI + l^2EA)\mathrm{w}'''' - l^2EI\mathrm{w}''''''\,. \quad (9.5)$$

The next steps depend on whether Hamilton's principle version (7.28), based on both classical and non - classical kinetic energies, or Hamilton's principle (7.34), based on kinetic energy only, is adopted.

9.1.1 Version 1: Hamilton's principle based on both classical and non - classical kinetic energies

We assume Eq. (7.28) to apply, with the total kinetic energy $\overline{T}$ given in Eqs. (7.25), (7.2) and (7.22). With **u** given in Eq. (5.162), we find that

$$T^{(cl.)} = \int_V \frac{1}{2}\rho \dot{u}_k \dot{u}_k dV = \int_V \frac{1}{2}\rho[(\dot{w}')^2 z^2 + \dot{w}^2]dV\,, \tag{9.6}$$

$$T^{(noncl.)} = \int_V \frac{1}{2}\gamma(\partial_p \dot{u}_k)(\partial_p \dot{u}_k)dV = \int_V \frac{1}{2}\gamma[(\dot{w}'')^2 z^2 + 2(\dot{w}')^2]dV\,, \tag{9.7}$$

from which,

$$\begin{aligned}\overline{T} &= T^{(cl.)} + T^{(noncl.)} \\ &= \int_0^L \frac{1}{2}\rho[I(\dot{w}')^2 + A\dot{w}^2]dx + \int_0^L \frac{1}{2}\gamma[I(\dot{w}'')^2 + 2A(\dot{w}')^2]dx\,.\end{aligned} \tag{9.8}$$

After lengthy manipulations, using repeatedly partial integration and divergence theorem, we arrive at

$$\begin{aligned}\delta\int_{t_1}^{t_2}\int_V \overline{T}\,dV\,dt = \int_{t_1}^{t_2}\{\,[\gamma I\ddot{w}''\delta(-w')]_{x=0}^{x=L} - \left[\left((\rho I + 2A\gamma)\ddot{w}' - \gamma I\ddot{w}'''\right)\delta w\right]_{x=0}^{x=L} \\ - \int_0^L[\rho A\ddot{w} - (\rho I + 2A\gamma)(\ddot{w}'' - \gamma I\ddot{w}'''')]\delta w\,dx\}dt\,.\end{aligned} \tag{9.9}$$

In analogy to section 7.2 and in order to emphasize that this approach relies upon total kinetic energy, we write $M^{(T)}$, $Q^{(T)}$ and $m^{(T)}$ instead of M, Q, and m, respectively. Then, substitution of formulas (9.1), (9.2) and (9.9) into Hamilton's principle (7.28),

$$\delta\int_{t_1}^{t_2}\left(T - \Pi^{(i)}\right)dt + \int_{t_1}^{t_2}\delta W^{(e)}dt = 0\,, \tag{9,10}$$

dropping the time integration and comparison of terms in the usual manner, gives the sectional constitutive equations

$$\gamma I\ddot{w}'' - f_1 + M^{(T)} = 0 \Leftrightarrow M^{(T)} = -(EI + l^2 EA)w'' + l^2 EIw'''' - \gamma I\ddot{w}'', \tag{9.11}$$

$$-f_2 + m^{(T)} = 0 \Leftrightarrow m^{(T)} = -l^2 EIw''', \tag{9.12}$$

the sectional field equation

$$-(\rho I + 2A\gamma)\ddot{w}' + \gamma I\ddot{w}''' - f_1' + Q^{(T)} = 0 \;\Leftrightarrow\; \left(M^{(T)}\right)' - Q^{(T)} = -(\rho I + 2A\gamma)\ddot{w}' \tag{9.13}$$

the governing equation of motion

$$\begin{aligned}-\rho A\ddot{w} + (\rho I + 2\gamma A)\ddot{w}'' - \gamma I\ddot{w}'''' - f_3 + q = 0 \;\Leftrightarrow \\ -\rho A\ddot{w} + (\rho I + 2\gamma A)\ddot{w}'' - \gamma I\ddot{w}'''' - (EI + l^2 EA)w'''' + l^2 EIw'''''' + q = 0\,,\end{aligned} \tag{9.14}$$

and the boundary conditions

$$\text{either } Q^{(T)} \text{ or w, either } M^{(T)} \text{ or } w' \text{ and} \tag{9.15}$$

$$\text{either } m^{(T)} \text{ or } w'' \tag{9.16}$$

have to be prescribed at $\mathrm{x} = 0, L$. Here, $M^{(T)}$ and $m^{(T)}$ are given in Eqs. (9.11), (9.12), while $Q^{(T)}$ can be determined from Eq. (9.13),

$$Q^{(T)} = -(EI + l^2 EA)\mathrm{w}''' + l^2 EI\mathrm{w}''''' - \gamma I\dot{\mathrm{w}}''' + (\rho I + 2\gamma A)\ddot{\mathrm{w}}'. \tag{9.17}$$

9.1.2 Version 2: Hamilton's principle based solely on classical kinetic energy

Assume now Hamilton's principle (7.34) to apply with $\hat{T} = T^{(cl.)}$ as in Eq. (7.2),

$$T^{(cl.)} = \int_V \frac{1}{2}\rho\dot{\mathrm{u}}_k\dot{\mathrm{u}}_k dV = \int_V \frac{1}{2}\rho\left[(\dot{\mathrm{w}}')^2\mathrm{z}^2 + \dot{\mathrm{w}}^2\right]dV \implies$$

$$\delta\int_{t_1}^{t_2} T^{(cl.)}dt = -\int_{t_1}^{t_2}\{\,[\rho I\ddot{\mathrm{w}}'\delta\mathrm{w}]_{\mathrm{x}=0}^{\mathrm{x}=L} - \int_0^L(\rho I\ddot{\mathrm{w}}'' - \rho A\ddot{\mathrm{w}})\delta\mathrm{w}\,d\mathrm{x}\}dt \tag{9.18}$$

and with

$$\int_{t_1}^{t_2}\delta\widehat{W}^{(noncl.inert.)}\,dt = \int_{t_1}^{t_2}\int_V \gamma\Delta\dot{\mathrm{u}}_k\,\delta\mathrm{u}_k dV\,dt$$

$$= \int_{t_1}^{t_2}\int_V \gamma[\dot{\mathrm{w}}'''\mathrm{z}^2\delta\mathrm{w}' + \dot{\mathrm{w}}''\delta\mathrm{w}]dV\,dt = \int_{t_1}^{t_2}\int_V[\gamma I\dot{\mathrm{w}}'''\delta\mathrm{w}' + \gamma A\dot{\mathrm{w}}''\delta\mathrm{w}]d\mathrm{x}\,dt$$

$$= \int_{t_1}^{t_2}\{\,[\gamma I\dot{\mathrm{w}}'''\delta\mathrm{w}]_{\mathrm{x}=0}^{\mathrm{x}=L} - \int_0^L(\gamma I\dot{\mathrm{w}}'''' - \gamma A\dot{\mathrm{w}}'')\delta\mathrm{w}\,d\mathrm{x}\}dt\,. \tag{9.19}$$

In analogy to section 7.3, we write $M^{(P)}$, $Q^{(P)}$, $m^{(P)}$ and $\delta\widehat{W}^{(e)}$ instead of M, Q, m and $\delta W^{(e)}$ in Eq. (9.1). Then, substitution of these equations and of Eqs. (9.1), (9.2) into Hamilton's principle (7.34), i.e.,

$$\delta\int_{t_1}^{t_2}\left(T^{(cl)} - \Pi^{(i)}\right)dt + \int_{t_1}^{t_2}\delta\widehat{W}^{(e)}\,dt + \int_{t_1}^{t_2}\delta\widehat{W}^{(noncl.inert.)}\,dt = 0\,, \tag{9.20}$$

dropping the time integration and comparison of terms, leads to the sectional constitutive laws

$$-f_1 + M^{(P)} = 0 \Leftrightarrow M^{(P)} = -(EI + l^2 EA)\mathrm{w}'' + l^2 EI\mathrm{w}''''\,, \tag{9.21}$$

$$-f_2 + m^{(P)} = 0 \;\Leftrightarrow\; m^{(P)} = -l^2 EI\mathrm{w}'''\,, \tag{9.22}$$

the sectional field equation

$$-\rho I\ddot{\mathrm{w}}' + \gamma I\dot{\mathrm{w}}''' - f_1' + Q^{(P)} = 0 \;\Leftrightarrow\; \left(M^{(P)}\right)' - Q^{(P)} = -\rho I\ddot{\mathrm{w}}' + \gamma I\dot{\mathrm{w}}'''\,, \tag{9.23}$$

the equation of motion

$$-\rho A\ddot{\mathrm{w}} + (\rho I + \gamma A)\ddot{\mathrm{w}}'' - \gamma I\ddot{\mathrm{w}}'''' - f_3 + q = 0 \qquad \Leftrightarrow$$

$$-\rho A\ddot{\mathrm{w}} + (\rho I + \gamma A)\ddot{\mathrm{w}}'' - \gamma I\ddot{\mathrm{w}}'''' - (EI + l^2 EA)\mathrm{w}'''' + l^2 EI\mathrm{w}'''''' + q = 0\,, \tag{9.24}$$

and the boundary conditions

$$\text{either } Q^{(P)} \text{ or w, either } M^{(P)} \text{ or w}' \text{ and} \tag{9.25}$$

$$\text{either } m^{(P)} \text{ or w}'' \tag{9.26}$$

have to be prescribed at $\mathrm{x} = 0, L$. Here, $M^{(P)}$ and $m^{(P)}$ are given in Eqs. (9.21), (9.22), while $Q^{(P)}$ can be determined from Eq. (9.23),

$$Q^{(P)} = -(EI + l^2 EA)\mathrm{w}''' + l^2 EI\mathrm{w}''''' - \gamma I\ddot{\mathrm{w}}''' + \rho I\ddot{\mathrm{w}}' \,. \tag{9.27}$$

9.2 Beam under dynamical transverse load

Consider the bar of Fig. 45, which is fixed at its left end and subject to concentrated dynamic transverse load at its right end, harmonically varying over time. The bar is of length L and sectional area A.

Figure 45: Cantilever beam subject to transverse load $\tilde{F}_0 e^{i\tilde{\omega}\tilde{t}}$.

The definitions of dimensionless variables in Eqs. (8.7), (8.8) hold, and in addition

$$\tilde{A} \coloneqq \frac{A}{L^2}\,, \quad \tilde{I} \coloneqq \frac{I}{L^4}\,, \quad \tilde{F}_0 = \frac{F_0}{EL^2}\,. \tag{9.28}$$

First, we consider the case of absent non - classical inertial terms, i.e., $\gamma = 0$. We can recast the governing equation for the KG - Model as follows (cf. Eq. (9.14))

$$-\rho A\ddot{\mathrm{w}} + \rho I\ddot{\mathrm{w}}'' - (EI + l^2 EA)\mathrm{w}'''' + l^2 EI\mathrm{w}'''''' = 0\,, \tag{9.29}$$

while the governing equation for the case of classical elasticity is

$$-\rho A\ddot{\mathrm{w}} + \rho I\ddot{\mathrm{w}}'' - EI\mathrm{w}'''' = 0\,. \tag{9.30}$$

Utilizing (8.7), (8.8) and (9.28), the dimensionless forms of these equations are

$$-\tilde{A}\ddot{\tilde{\mathrm{w}}} + \tilde{I}\ddot{\tilde{\mathrm{w}}}'' - \left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\tilde{\mathrm{w}}'''' + \tilde{l}^2\tilde{I}\tilde{\mathrm{w}}'''''' = 0 \quad \text{(KG - Model)}\,, \tag{9.31}$$

$$-\tilde{A}\ddot{\tilde{\mathrm{w}}} + \tilde{I}\ddot{\tilde{\mathrm{w}}}'' - \tilde{I}\tilde{\mathrm{w}}'''' = 0 \quad \text{(classical elasticity)}\,. \tag{9.32}$$

The classical boundary condition at $\tilde{\mathrm{x}} = 1$ will be a harmonically varying force $\tilde{F}_0 e^{i\tilde{\omega}\tilde{t}}$, with $\tilde{\omega}$ being a dimensionless operating frequency, $\tilde{F}_0$ being the force amplitude in dimensionless form and i being the imaginary unit. This kind of loading condition suggests assuming for the solutions of Eqs. (9.31), (9.32) to be of the form

$$\widetilde{\mathrm{w}}(\tilde{\mathrm{x}},\tilde{t}) = \widetilde{\mathrm{W}}(\tilde{\mathrm{x}})e^{i\widetilde{\omega}\tilde{t}}\,. \tag{9.33}$$

After substitution into (9.31), (9.32) and elimination of the factor $e^{i\widetilde{\omega}\tilde{t}}$, we obtain

$$\tilde{A}\widetilde{\omega}^2\widetilde{\mathrm{W}} - \tilde{I}\widetilde{\omega}^2\widetilde{\mathrm{W}}'' - \left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}'''' + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}'''''' = 0 \qquad \text{(KG - Model)}\,, \tag{9.34}$$

$$\tilde{A}\widetilde{\omega}^2\widetilde{\mathrm{W}} + \tilde{I}\widetilde{\omega}^2\widetilde{\mathrm{W}}'' - \tilde{I}\widetilde{\mathrm{W}}'''' = 0 \qquad \text{(classical elasticity)}\,. \tag{9.35}$$

We assume non - classical tractions to vanish at both ends of the bar and hence the imposed boundary conditions are of the form

$$\mathrm{w}(0) = 0\,,\ \ \mathrm{w}'(0) = 0\,,\ \ M(L) = 0\,,\ \ Q(L) = F_0 e^{i\omega t}\,,\ \ m(0) = m(L) = 0\,. \tag{9.36}$$

With the aid of Eqs. (9.11), (9.12), (9.17), for $\gamma = 0$, the definitions (8.7), (8.8) and (9.28), and after eliminating the factor $e^{i\widetilde{\omega}\tilde{t}}$, we can recast the boundary conditions for the KG - Model as

$$\widetilde{\mathrm{W}}(0) = 0\,, \qquad \widetilde{\mathrm{W}}'(0) = 0\,, \qquad -\left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}''(1) + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}''''(1) = 0\,, \tag{9.37}$$

$$-\left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}'''(1) + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}'''''(1) - \widetilde{\omega}^2\tilde{I}\widetilde{\mathrm{W}}'(1) = \tilde{F}_0\,, \tag{9.38}$$

$$\widetilde{\mathrm{W}}'''(0) = 0\,, \qquad \widetilde{\mathrm{W}}'''(1) = 0\,, \tag{9.39}$$

and for the classical elasticity as

$$\widetilde{\mathrm{W}}(0) = 0\,, \qquad \widetilde{\mathrm{W}}'(0) = 0\,, \qquad \widetilde{\mathrm{W}}''(1) = 0\,, \qquad -\tilde{I}\widetilde{\mathrm{W}}'''(1) - \widetilde{\omega}^2\tilde{I}\widetilde{\mathrm{W}}'(1) = \tilde{F}_0\,. \tag{9.40}$$

Numerical solutions of the above equations for the KG - Model and classical elasticity are illustrated in Fig. 46. In the calculations we have set $\tilde{A} = 0.004$, $\tilde{I} = 5.5 \cdot 10^{-6}$ and we have chosen the constant value $\tilde{F}_0 = 5 \cdot 10^{-9}$.

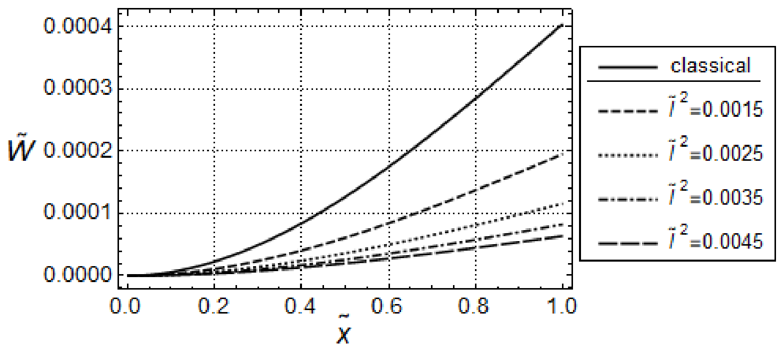


Figure 46: Distributions of $\widetilde{\mathrm{W}}$ for the KG - Model and classical elasticity for various values of $\tilde{l}^2$ and for frequency $\widetilde{\omega} = 0.05$.

It can be seen, that for the sufficiently small frequency $\widetilde{\omega} = 0.05$, the distributions of $\widetilde{\mathrm{W}}$ predicted by the KG - Model indicate the gradient stiffening effect in comparison to the classical solution. The stiffening effect increases with increasing values of $\tilde{l}$. This behaviour is quite similar to the dynamic axial loading (cf. examples discussed in section 8.2, Fig. 32).

Next, we consider the case of present non - classical inertial terms, i.e., $\gamma \neq 0$. As above, there are two versions. First, both classical and non - classical kinetic energies are taken into consideration (see section 9.1.1) and secondly, we deal only with classical kinetic energy (see section 9.1.2).

Version 1

For the particular case where both classical and non - classical kinetic energies are present, the governing equation for the KG - Model is (see Eq. (9.14))

$$-\rho A\ddot{\mathrm{w}} + (\rho I + 2\gamma A)\ddot{\mathrm{w}}'' - \gamma I\ddot{\mathrm{w}}'''' - (EI + l^2 EA)\mathrm{w}'''' + l^2 EI\mathrm{w}'''''' = 0\,, \tag{9.41}$$

and the imposed boundary conditions are

$$\mathrm{w}(0) = 0\,,\ \ \mathrm{w}'(0) = 0\,,\ \ M^{(T)}(L) = 0\,,\ \ Q^{(T)}(L) = F_0 e^{i\omega t}\,,\ \ m^{(T)}(0) = m^{(T)}(L) = 0\,. \tag{9.42}$$

With the aid of definitions (8.7), (8.8), (9.28) and Eq. (9.33) we can rewrite the governing equation as

$$\tilde{A}\tilde{\omega}^2\widetilde{\mathrm{W}} - \left(\tilde{I} + 2\tilde{\gamma}\tilde{A}\right)\tilde{\omega}^2\widetilde{\mathrm{W}}'' + \left[\tilde{I}(\tilde{\gamma}\tilde{\omega}^2 - 1) - \tilde{l}^2\tilde{A}\right]\widetilde{\mathrm{W}}'''' + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}'''''' = 0\,, \tag{9.43}$$

and the concomitant boundary conditions as

$$\widetilde{\mathrm{W}}(0) = 0\,,\ \ \widetilde{\mathrm{W}}'(0) = 0\,,\ \ -\left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}''(1) + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}''''(1) + \tilde{\gamma}\tilde{\omega}^2\tilde{I}\widetilde{\mathrm{W}}''(1) = 0\,, \tag{9.44}$$

$$-\left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}'''(1) + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}'''''(1) + \tilde{\gamma}\tilde{\omega}^2\tilde{I}\widetilde{\mathrm{W}}'''(1) - \left(\tilde{I} + 2\tilde{\gamma}\tilde{A}\right)\tilde{\omega}^2\widetilde{\mathrm{W}}'(1) = \tilde{F}_0\,, \tag{9.45}$$

$$\widetilde{\mathrm{W}}'''(0) = 0\,,\ \ \widetilde{\mathrm{W}}'''(1) = 0\,. \tag{9.46}$$

Version 2

Lastly, for the particular case where the kinetic energy consists only of classical part, the governing equation for the KG - Model is (see Eq. (9.24))

$$-\rho A\ddot{\mathrm{w}} + (\rho I + \gamma A)\ddot{\mathrm{w}}'' - \gamma I\ddot{\mathrm{w}}'''' - (EI + l^2 EA)\mathrm{w}'''' + l^2 EI\mathrm{w}'''''' + q = 0\,, \tag{9.47}$$

and the imposed boundary conditions are

$$\mathrm{w}(0) = 0\,,\ \ \mathrm{w}'(0) = 0\,,\ \ M^{(P)}(L) = 0\,,\ \ Q^{(P)}(L) = F_0 e^{i\omega t},\ \ m^{(P)}(0) = m^{(P)}(L) = 0\,. \tag{9.48}$$

Once again, using similar manipulations as before, we can recast the governing equation as

$$\tilde{A}\tilde{\omega}^2\widetilde{\mathrm{W}} - \left(\tilde{I} + \tilde{\gamma}\tilde{A}\right)\tilde{\omega}^2\widetilde{\mathrm{W}}'' + \left[\tilde{I}(\tilde{\gamma}\tilde{\omega}^2 - 1) - \tilde{l}^2\tilde{A}\right]\widetilde{\mathrm{W}}'''' + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}'''''' = 0\,, \tag{9.49}$$

and the concomitant boundary conditions as

$$\widetilde{\mathrm{W}}(0) = 0\,,\ \ \ \widetilde{\mathrm{W}}'(0) = 0,\ \ \ -\left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}''(1) + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}''''(1) = 0\,, \tag{9.50}$$

$$-\left(\tilde{I} + \tilde{l}^2\tilde{A}\right)\widetilde{\mathrm{W}}'''(1) + \tilde{l}^2\tilde{I}\widetilde{\mathrm{W}}'''''(1) - \tilde{\omega}^2\tilde{I}\widetilde{\mathrm{W}}'(1) + \tilde{\gamma}\tilde{\omega}^2\tilde{I} + 2\widetilde{\mathrm{W}}'''(1) = \tilde{F}_0\,, \tag{9.51}$$

$$\widetilde{\mathrm{W}}'''(0) = 0\,,\ \ \ \widetilde{\mathrm{W}}'''(1) = 0\,. \tag{9.52}$$

Clearly, the solutions for both Version 1 and Version 2 will also be of the form (9.33).

Distributions of the amplitude $\widetilde{\mathrm{W}}$ of the deflection are dislpayed in Fig. 47 for both versions and for classical elasticity.

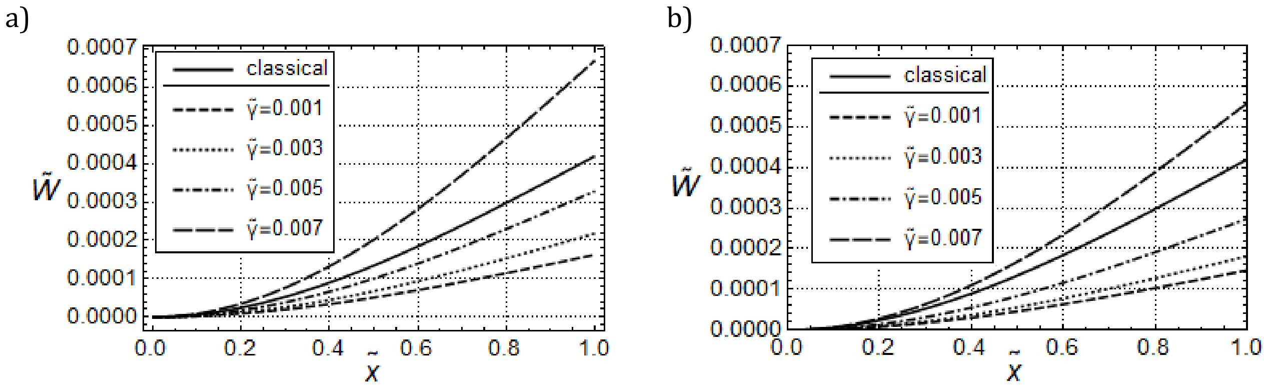


Figure 47: Distributions $\widetilde{W}$ for a) Version 1 and b) Version 2 of gradient elasticity, for various values of $\tilde{\gamma}$ and for $\tilde{l}^2 = 0.0025$ and frequency $\tilde{\omega} = 0.05$.

It is apparent that for the sufficiently small frequency $\tilde{\omega} = 0.05$ and for keeping $\tilde{l}$ constant, by increasing values of $\tilde{\gamma}$, the $\widetilde{W}$ - distributions predicted by both versions of the KG - Model are increasing and can exceed the one predicted by classical elasticity, which indicates that gradient softening effect can occur. In addition, Version 2 predicts smaller distributions than Version 1. This behaviour is also quite similar to the one observed for axial dynamic loading (examples discussed in section 8.3, Figs. 36, 40). Furthermore, the relationships for higher frequencies are similar for both one - dimensional axial loading and bending under concentrated transverse loading. Therefore, responses for higher values of $\tilde{\omega}$ are not discussed further. It is only remarked that certain patterns of responses are observable for smaller values of $\tilde{\omega}$ for the bending loading problems than for the axial loading problems.

Similar to the one - dimensional examples, the plausibility arguments of the predicted responses do not allow to reject the one version over the other. However, Version 1 does not satisfy the principle of material frame indifference and cannot be physically accepted.

10 Concluding remarks

The present thesis is concerned with a simple model of explicit gradient elasticity, denoted as KG - Model. This model can be established as a particular case of Mindlin's gradient elasticity or as a gradient counterpart of Kelvin's viscoelastic solid. In the second case, the appropriate thermodynamics framework is provided by the non - equilibrium thermodynamics introduced by Alber et al. [2]. The thesis highlights specific properties of the KG - Model in statics and dynamics. Especially, for every problem examined, a comparison is made between the KG - Model and the classical elasticity. The investigations themselves are mainly performed analytically, so closed - form solutions are elaborated. The discussions refer to one - dimensional problems with axial loading and two - dimensional bending problems. In what concerns the one - dimensional problems in statics, our attention is focused on the effect of different loading conditions and on the effect of internal material lengths inherent in the constitutive law. It is shown that for monotonic loading the so - called gradient stiffening effect is occurring and that for vanishing internal material length, the responses converge to the one predicted by classical elasticity. In what concerns bending loading problems in statics, the aim of this work was threefold.

1. To elaborate and to examine a consistent Euler - Bernoulli beam theory, for both classical and gradient elasticity.

2. Having a consistent Euler - Bernoulli beam theory available, to derive stress distributions for bending loading, besides deflection curves.

3. To employ for the first time a new approach to buckling problems. For vanishing non - classical boundary conditions, this buckling approach leads to the same critical load obtained commonly in literature of gradient elasticity. However, if certain non - classical boundary conditions do not vanish, then the results according to the approach of the thesis are different from the ones according to usual gradient elasticity.

The discussion of problems in dynamics also refers to one - dimensional axial and two - dimensional bending loading conditions. Here, emphasis is given on whether non - classical inertial terms are present or not and the effect thereof. The discussions are based on appropriately derived versions of Hamilton's principle. One version draws back to Mindlin and incorporates inertial terms in the boundary tractions. A second version, due to Broese et al. [8, 9] leads to boundary tractions which do not include inertial terms. Counterparts of these versions for Euler - Bernoulli beam theory have been developed in [34] and are utilized in this thesis. The differences between these two versions have been illustrated with the aid of one - dimensional axial and two - dimensional bending examples. The established results do not allow to prefer one version over the other on the basis of plausibility arguments. Nevertheless, Version 1 cannot be physically accepted as it violates the principle of material frame indifference.

11 Appendix A

(Section forces for one - dimensional tension - compression loading conditions).

Assume the beam of section 5.2.C) subject to tension - compression loading with

$$\mathrm{u}_i = \mathrm{u}_i(\mathrm{x}) \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{u} \\ 0 \\ 0 \end{pmatrix} \Rightarrow \delta\mathbf{u} \mathrel{\widehat{=}} \begin{pmatrix} \delta\mathrm{u} \\ 0 \\ 0 \end{pmatrix}, \tag{11.1}$$

$$D\mathrm{u}_k = \mathrm{n}_l\, \partial_l \mathrm{u}_k \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{n}_l\, \partial_l\, \mathrm{u} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathrm{n}_l \mathrm{u}' \\ 0 \\ 0 \end{pmatrix} \Rightarrow \delta(D\mathrm{u}_k) \mathrel{\widehat{=}} \begin{pmatrix} \mathrm{n}_l \delta\mathrm{u}' \\ 0 \\ 0 \end{pmatrix}. \tag{11.2}$$

There is only one non - vanishing strain component, $\epsilon = \mathrm{u}'$, and the only non - vanishing stress components are Σ_{11} and μ_{111}. Therefore, on section planes A ($\mathbf{n} = \mathbf{e}_\mathrm{x}$) we find from Eq. (5.175) that

$$[P_i]_A \mathrel{\widehat{=}} \mathrm{n}_1 \begin{pmatrix} \Sigma_{11} \\ 0 \\ 0 \end{pmatrix}, \tag{11.3}$$

$$[R_i]_A \mathrel{\widehat{=}} \begin{pmatrix} \mu_{111} \\ 0 \\ 0 \end{pmatrix}. \tag{11.4}$$

The virtual work expended by **P**, **R** on A is given by

$$\int_A \mathbf{P} \cdot \delta\mathbf{u} dS + \int_A \mathbf{R} \cdot \delta(D\mathbf{u}) dS = \left[\int_A P_1 dS\right] \delta\mathrm{u} + \left[\int_A R_1 dS\right] \mathrm{n}_1 \delta\mathrm{u}'. \tag{11.5}$$

This suggests the definition of section classical normal force vector

$$\mathbf{N} = \mathbf{N}(\mathbf{n}, \mathrm{x}) = N \mathrm{n}_1 \mathbf{e}_\mathrm{x} \equiv N\mathbf{n}, \tag{11.6}$$

$$N = N(\mathrm{x}) := \int_A P_1 dS, \tag{11.7}$$

and of section non - classical normal force vector

$$\mathbf{N}_R = \mathbf{N}_R(\mathrm{n,x}) = N_R \mathrm{n}_1 \mathbf{e}_\mathrm{x} \equiv N_R \mathbf{n}, \tag{11.8}$$

$$N_R = N_R(\mathrm{x}) := \int_A R_1 dS, \tag{11.9}$$

so that

$$\int_A \mathbf{P} \cdot \delta\mathbf{u} dS + \int_A \mathbf{R} \cdot \delta(D\mathbf{u}) dS = N \mathrm{n}_1 \delta\mathrm{u} + N_R \mathrm{n}_1 \delta\mathrm{u}'. \tag{11.10}$$

12 Appendix B

The differential equation (5.286) is a linear nonhomogeneous sixth - order differential equation of the form

$$\alpha y''''(x) - \beta y''''''(x) = q\,, \tag{12.1}$$

where, in our case,

$$\alpha = 1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2, \qquad \beta = \tilde{l}^2\,, \qquad q = \frac{3\tilde{q}_0}{4\tilde{b}\tilde{c}^3}\,. \tag{12.2}$$

The solution of this differential equation is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + c_5e^{\sqrt{\frac{\alpha}{\beta}}x} + c_6e^{-\sqrt{\frac{\alpha}{\beta}}x} + \frac{q}{24\alpha}x^4\,, \tag{12.3}$$

with constants of integration being

$$c_1 = \frac{\beta^2 q\sqrt{\frac{\alpha}{\beta}}\left(e^{2\sqrt{\frac{\alpha}{\beta}}} + 1\right)}{\alpha^3\left(e^{\sqrt{\frac{\alpha}{\beta}}} - 1\right)\left(e^{\sqrt{\frac{\alpha}{\beta}}} + 1\right)}\,, \tag{12.4}$$

$$c_2 = -\frac{\beta q}{\alpha^2}\,, \tag{12.5}$$

$$c_3 = \frac{q(\alpha + 2\beta)}{4\alpha^2}\,, \tag{12.6}$$

$$c_4 = -\frac{q}{6\alpha}\,, \tag{12.7}$$

$$c_5 = -\frac{\beta^2 q\sqrt{\frac{\alpha}{\beta}}}{\alpha^3\left(e^{\sqrt{\frac{\alpha}{\beta}}} - 1\right)\left(e^{\sqrt{\frac{\alpha}{\beta}}} + 1\right)}\,, \tag{12.8}$$

$$c_6 = -\frac{\beta^2 q\sqrt{\frac{\alpha}{\beta}}e^{2\sqrt{\frac{\alpha}{\beta}}}}{\alpha^3\left(e^{\sqrt{\frac{\alpha}{\beta}}} - 1\right)\left(e^{\sqrt{\frac{\alpha}{\beta}}} + 1\right)}\,. \tag{12.9}$$

13 Appendix C

The components a_{11} , ..., a_{44}, (see Eq. (6.35), p. 68) are

$$a_{11} = -\tilde{\xi}^3\,, \;\; a_{12} = 0\,, \;\; a_{13} = \tilde{\theta}^3\,, \;\; a_{14} = 0\,, \;\; a_{21} = 0\,, \;\; a_{22} = -\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\xi}^2 - \tilde{l}^2\tilde{\xi}^4\,,$$

$$a_{23} = 0\,, \;\; a_{24} = \left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\theta}^2 - \tilde{l}^2\tilde{\theta}^4\,, \;\; a_{31} = -\tilde{\xi}^3\cos\tilde{\xi}, \;\; a_{32} = \tilde{\xi}^3\sin\tilde{\xi}\,,$$

$$a_{33} = \tilde{\theta}^3\cosh\tilde{\theta}\,, \;\; a_{34} = \tilde{\theta}^3\sinh\tilde{\theta}, \;\; a_{41} = -\left[\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\xi}^2 + \tilde{l}^2\tilde{\xi}^4\right]\sin\tilde{\xi}\,,$$

$$a_{42} = -\left[\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\xi}^2 + \tilde{l}^2\tilde{\xi}^4\right]\cos\tilde{\xi}\,, \;\; a_{43} = \left[\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\theta}^2 - \tilde{l}^2\tilde{\theta}^4\right]\sinh\tilde{\theta}\,,$$

$$a_{44} = \left[\left(1 + 3\left(\frac{\tilde{l}}{\tilde{c}}\right)^2\right)\tilde{\theta}^2 - \tilde{l}^2\tilde{\theta}^4\right]\cosh\tilde{\theta}, \tag{13.1}$$

with $\tilde{\xi}$ and $\tilde{\theta}$, defined in Eqs. (6.32) and (6.33), being functions of $\tilde{k}$.

Table of figures

List of abbreviations

cf.	compare
cl.	classical
cons.	conservative
const.	constant
cr	critical
e.g.	for example
Eq(s).	equation(s)
ext.	external
Fig.	figure
KG	Kelvin Gradient
i.e.	that is
inert.	inertial
N.A.	Neutral Axis
noncl.	nonclassical
noncons.	nonconservative
(i)	internal
(e)	external

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